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An Eshelbian Micromechanics Approach to Non-saturated Porous Media

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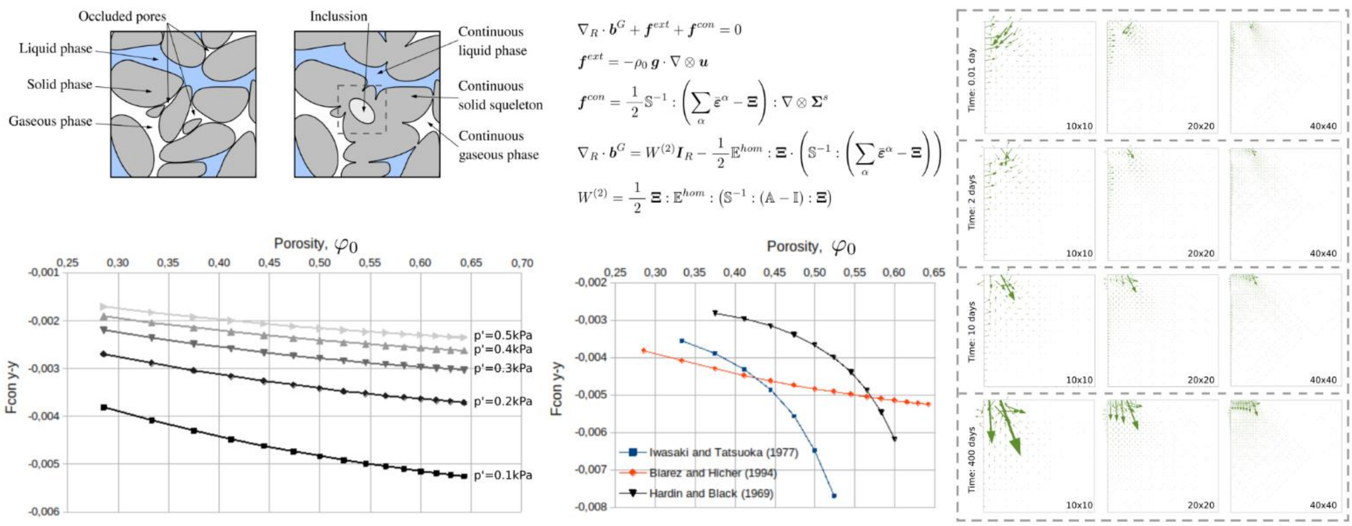
Abstract
The main goal of the present paper is to develop a mathematical framework for modeling the field equations arising in the problem of an elastic, isotropic, non-saturated porous media (pores filled with air and water) within the context of Eshelbian mechanics. The global balance of pseudomomentum is performed in a fully material manifold to account for the configurational forces due to material inhomogeneities, involving the Maxwell stress tensor. Biot's momentum conservation equations in a dilute scheme for a micromechanical environment, combined with the Mori–Tanaka homogenization theory, are employed for the geomechanical solution. In the mathematical description, pores are treated as Eshelby inhomogeneous inclusions within a solid skeleton, making them the source of configurational forces. The resulting field equations show that these configurational forces evolve in a strongly nonlinear manner due to their dependence on the nature of the pores as well as the soil's mechanical properties. This behavior was numerically observed through the implementation of the boundary value problem using the Finite Element Method

Keywords
Mori-Tanaka Method, Micromechanics, Porous media, Eshelbian mechanics

Graphical Abstract



1 INTRODUCTION

Eshelbian mechanics or configurational mechanics may be regarded as a subset of the theory of material inhomogeneities, namely, when certain material properties such as density and elasticity coefficient, undergo continuum variations even without external loadings. It is a discipline that mainly addresses with a special type of force, called configurational force (in contrast to physical forces that are the structural response to an actual displacement of a material particle), that allows these inhomogeneities to be handled as defects such as inclusions, dislocations, fractures or more generally, a sudden change at a certain material point without, as mentioned, external actions (surface tractions, mass force, etc.). Any material inhomogeneity may be ensued by a translation of the physical system, i.e. a pullback of the ordinary balance equation, on the material manifold (Steinmann, 2015). The resulting system of equations is called the balance of pseudomomentum or simply the unbalanced equation (Maugin, 1993) because it is in fact a balance equation in the absence of an external load but is unbalanced by inhomogeneity. These equations, along with the fundamental Noether's theorem have become one of the most versatile branches of mechanics feasible for extension to the general theory of fields (Maugin, 2017).

The birth of a true configurational mechanic (Eshelbian mechanics) stems from Eshelby's fundamental work (Eshelby, 1951) as well as Kröner´s work (Kröner & Datta 1966). The Eshelbian mechanics tenet hinges on two concepts, the abovementioned configurational forces along with the Eshelby/Maxwell stress tensor (Eshelby, 1951). Initially connected to the field of material uniformity, Maugin (Maugin, 1993) revisited the fundamental connections between the Maxwell stress tensor and the variational principles thereby fostering the mathematical formulation of different fields namely electromagnetic materials and fracture, geometrical aspects of elasticity (Steinmann, 2015), material growth (Gurtin 1999; Maugin, 2010), finite element solutions (Maugin, 2010), or, in short, the formulation of a considerable portion of classical and nonclassical mechanics in terms of Eshelby stress (Maugin, 2017)

From the pioneering work by Biot in poroelastic bodies (Biot, 1941) to complex and robust approaches, a broad range of mechanical situations, thermal conditions, fluid transport, boundary conditions and load types were considered. Biot himself extended his work to wave propagation (Biot, 1596). Two-phase and three-phase non-saturated cases were investigated in a continuous porous media theory and by Lewis and Schrefler (1998). Mroginski et al. (2010) described an odd relationship between the vertical displacement and the degree of pollutant saturation. The environmental geo-mechanics problem was addressed by Schrefler (2001). Beneyto et al. (2015) presented a different approach for this issue based on the stress state decomposition technique (SSDT), and in Di Rado et al. (2020), this same technique was extended to biological fields.

The solid phase constituent was regarded as elastic for the present scope; however, straightforward generalization to nonlineal models is possible considering the micromechanical strategy. Regarding to the former for partially saturated porous media, many authors have proposed approaches to predict macroscopic behavior based on the knowledge of some variables of the microstructure, such as the size of the Representative Volume Element (RVE), characteristic internal length, pore pressure, pore shape and distribution, among others. Generally, those solutions are mostly based on numerical homogenization techniques such as Squared Finite Element Method (FEM^2), Variational Asymptotic Method of Homogenization, or incremental techniques with different stages of homogenization. However, there still exists a theoretical gap in modeling the behavior of partially saturated heterogeneous porous media with inclusions in the solid matrix. The aim in obtaining a theoretical formulation for materials with these physical characteristics goes beyond the conceptual level of addressing the issue of occluded pores observed in the theory of continuous porous media (see the idealization assumed in Figure 1) but also in providing an accurate tool to predict the behavior of multiporosity materials such as bones, fractured rocks, or multiphase soils.

2 ESHELBIAN MECHANICS.

2.1 Configurational forces in infinitesimal strain theory

The configurational problem is usually formulated using finite strain theory (Maugin, 2010). However, it can be straightforwardly extended to infinitesimal strain theory. Following Eischen and Herrmann (1987), the pseudo momentum equation would be written as

$$ {}_{R}.b \quad f^{ext} \quad f^{con} \quad 0 \tag{1} $$

$$ b = \mathrm{W}\mathbb{I}_{R} \quad {}^{S}. \quad \mathrm{u} \tag{2} $$

$$f^{ext} \quad \rho_0 f. \quad \mathrm{u} \tag{3}$$

$$f^{con} = -\frac{\bar{W}}{X_n}\bigg|_{\exp l} \tag{4}$$

Where Σ^S is the external (or macro, distant from the inclusion) stress state and u the kinematic field of displacement, b is the elastic energy momentum tensor (Maxwell tensor or quasiestatic Eshelby tensor), $\rho_0 f$ body force, W is the potential energy, I_R is the unitary second order tensor, and f^{con} are the configurational forces.

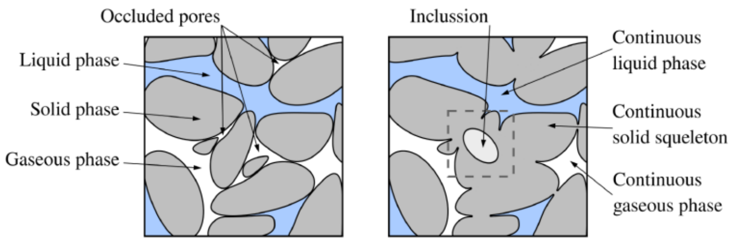


Figure 1 *Material description, classical micro hypothesis (left) and Continuous Porous Media Theory (right)*

2.2 Equivalent strain for inhomogeneous inclusion. Configurational forces and interaction energy

In poroelastic continuous soils, it is possible to treat voids in the same fashion as in the case of ellipsoidal inhomogeneity inclusions, i.e., voids filled with materials with elastic properties different from those of the soil grains or matrix. Eshelby (1957) coped with the general ellipsoidal inclusion problem and classified it as homogeneous (misfitting geometry) or inhomogeneous (misfitting properties), being the situation here considered, belonging to the second group.

Succinctly, the Eshelby proposal was that an ellipsoidal geometrical misfit in the matrix may be accounted for, through the interaction of two strain fields: a transformation strain (eigenstrain), ε^*, and a cancelation strain field, e^C Both strain fields are uniform in the inclusion (the stress inside the inclusion is also uniform), and they are related via a fourth-order tensor called the Eshelby fourth-order tensor, $\mathbb{S}$ (Eshelby, 1957; Mura, 1982)

$$e^C = \mathbb{S} : \varepsilon^* \tag{5}$$

Eshelby proposed treating the strains that arise due to misfitting properties as equivalent strains, where the term 'equivalent' refers to a comparison with the strains in the geometrical misfit problem—that is, an inhomogeneous inclusion treated as a homogeneous equivalent. For this purpose, in Figure 2, $\mathbb{E}$ denotes the elasticity tensor of the matrix and $\mathbb{E}$ that of the inclusion.

$$e^C = \mathbb{S} : \varepsilon_T^* \tag{6}$$

In Alhasadi & Federico (2017), a detailed description of both cases was carried out, and for the sake of brevity, only the strain state that arises in the inhomogeneous case is presented.

It can be inferred from Figure 2 that the cancelling strain e^C is a consequence of the elastic relaxation of the matrix when the surface tractions are released, and ε_T^* is a fictitious or equivalent transformation strain that, for the case in point, plays the role of a hypothetical real transformation strain in the case of inclusions with geometrical misfit.

Since the misfit is not geometrical, in the absence of an external stress state, no perturbation will be observed. However, with an external imposed stress field, the misfit will perturb it clamming for a correction.

Alhasadi & Salvatore (2017) derived a set of relationships of crucial importance for the scope of the present work. First, the cancelling strain is related to the external strain field or macro strain field (this concept will be thoroughly described hereinafter) through

$$e^{C} = (\mathbb{A} - \mathbb{II}) : \Xi \quad (7)$$

Where $\mathbb{A}$ is the strain concentration tensor in the inclusion (Hill, 1963) [41], $\mathbb{II}$ is the unitary fourth-order tensor, and Ξ is the strain field caused by the external or macro stress state. Using the previous in equation (6), leads to

$$\varepsilon_T^* = \mathbb{S}^{-1} : e^{C} = \mathbb{S}^{-1} : (\mathbb{A} - \mathbb{II}) : \Xi \quad (8)$$

Where $\mathbb{S}$ is the aforementioned Eshelby fourth-order tensor. In the following sections, simple expressions for both $\mathbb{A}$ and $\mathbb{S}$ are presented.

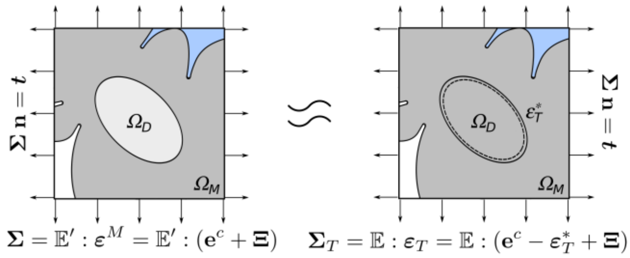


Figure 2 *Actual stress state (left) and the equivalent state (right)*

The most appropriate form for relating the concept of configurational force to the equivalent strain is based on the interaction energy. This concept was introduced in (Balluffi, 2012) and describes the interaction between the strain field due to the external load, namely the macro stress field (as well as macro strain, this concept will be described hereinafter), and the stress field around the inclusion. Alhasadi and Salvatore (2017), making use of the interaction energy concept, present the following expression for this energy

$$W^{\text{int}} = -\frac{1}{2} \int_D \Sigma^S : \varepsilon_T^* \quad (9)$$

Where W^{int} is the interaction energy. Hence, the latter in Eq. (4) along with Eq. (8), and replacing the explicit derivative of the interaction energy (around the fictitious transformation strain) by the limit around a vanishing quantity specially chosen, a very useful expression for the integral form of the configurational forces was obtained

$$F^{con} \quad \frac{1}{2} \int_D (\mathbb{S}^{-1} : (\mathbb{A} \quad \mathbb{II}) : \quad) : \quad {}^{s} \quad (10)$$

3 NONSATURATED MICROPOROMECHANICS.

3.1 Biot's Problem

The well-known solution to soil consolidation developed by M. Biot (Biot, 1941) may be briefly described by means of a system of coupled equations, namely, one equation of momentum conservation, one (or more) equation of fluid transport, one equation of mass conservation (occasionally through volume balance) and constitutive equations for a poroelastic-type solid and for fluids. In the present paper, no reference whatsoever will be pointed out to any transport equation restricting our scope to momentum conservation. As mentioned, the media is a poroelastic domain consisting of a group of pores (taking the place of inclusions), filled with water and air under pressure (a prestress state in the inclusion) and with stress interactions between these pores. One important purpose of the present paper is to cope with the isolated pores that arise in the Biot's problem and regarding these pores as stressed inclusions filled with water (liquid) or air (gas). For the sake of brevity, no further details of the Biot theory are given herein (see Lewis and Schrefler, 1998; Beneyto et al., 2015; Mroginski et al., 2011)

3.2 The dilute scheme for spherical pores. The Eshelby fourth-order and the strain concentration tensors.

Dormieux et al. (2006) noted that the dilute scheme is correct when the inclusions (pores) are small and sporadic, i.e., when developing a set of noninteracting units and a uniform microscopic strain tensor throughout the inclusion (pore) is satisfied; in line with the material hypothesis assumed in Figure 1. The general relationship between the local microscopic strain tensor and macroscopic strain tensor is generally formulated in the following manner (Hill, 1963)

$$\varepsilon^{M} = \mathrm{A} : \Xi \qquad (11)$$

Where ε^{M} is the micro strain in the inclusion. The dilute scheme allows one to rephrase local general relationships using average quantities in the same way as local quantities. Furthermore, this possibility is especially valuable when a further extension to Biot's problem is proposed. Basically, the average inclusion strain $\bar{\varepsilon}^{P}$ for the inclusion space (pore space) may be deemed from a single spherical inclusion ε^{SP}. Although it is not the final scope of this paper, it is a convenient starting point for introducing the main variables and concepts

$$\varepsilon^{SP} \quad \bar{\varepsilon}^{P} \quad \bar{\mathrm{A}}^{P} : \qquad (12)$$

Where $\bar{\mathrm{A}}^{P}$ is the mean strain concentration tensor for the pore space, $\bar{\varepsilon}^{P}$ is the average micropore strain (dropping inclusion reference). One main concern within the dilute scheme framework is determining the mean strain concentration tensor. In the precedent definition, the idea of the "mean" may be lawn down considering a medium with multiple pore domains. In this case, the average strain concentration tensor is defined by

$$\bar{\mathrm{A}}^{P} = \varphi_0^i \big/ \varphi_0 \, \bar{\mathrm{A}}^{i} \qquad (13)$$

where $\varphi_0 = \Omega^P / \Omega$ is the porosity in the volume fraction (as abovementioned, the volume fraction must be at most $\varphi_0 << 1$), Ω^P is the pore space, Ω is the whole space and φ_0^i is the porosity of each different pore domain $\varphi_0 \quad {}_i\,\varphi_0^i$

The average strain concentration tensor is a cornerstone concept for the assessment of many of the poroelastic constants. For example, the following expression for the homogenized stiffness tensor $\mathrm{E}^{\mathrm{hom}}$ (Mura, 1982)

$$\mathrm{E}^{\mathrm{hom}} = \mathrm{E}^{s} : (\mathrm{II} - \varphi_0 \bar{\mathrm{A}}^{P}) \qquad (14)$$

Where E^{s} constitutive tensor for the solid matrix, is based on the mean value of the strain concentration tensor, $\bar{\mathrm{A}}^{P}$. From the previous, a concise form for $\bar{\mathrm{A}}^{P}$ may be derived. According to Dormieux et al. (2006), for an empty porous medium, the average strain in an elliptic/spheric pore (SP) can be estimated as

$$\varepsilon^{SP} \cong \bar{\varepsilon}^{P} = (\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1} : \Xi \qquad (15)$$

Where G is the fourth order Hill polarization tensor (Barnett & Cai, 2018) relating pore stress and microstrain (without discerning between pore or solid structure provided that the microstrain is uniform throughout the domain) $\varepsilon^{SP} = -\mathrm{G} : \Sigma^{SP} + \Xi$; $\delta\mathrm{E} = \mathrm{E}^{SP} - \mathrm{E}^{S}$ is the difference between the pore and the solid elastic stiffness, and $\Sigma^{SP} = \delta\mathrm{E} : \varepsilon^{SP}$. For the point case, i.e., empty pores, the elastic stiffness is zero leading to $\delta\mathrm{E} = -\mathrm{E}^{S}$ (it is worthwhile assuming that E^{SP} is, in general, negligible). Furthermore, a simple version of the fourth-order Eshelby tensor is introduced (Dormieux et al, 2006)

$$\mathrm{S} = \mathrm{G} : \mathrm{E}^S \tag{16}$$

the former expression for pore strain may be rewritten as

$$\varepsilon^{SP} \cong \bar{\varepsilon}^P = (\mathrm{II} - \mathrm{S})^{-1} : \Xi \tag{17}$$

With these and accepting the aforesaid inference of Eq (12), the expression for the average strain concentration tensor for empty pores in the dilute scheme reads:

$$\bar{\mathrm{A}}^P = (\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1} = (\mathrm{II} - \mathrm{S})^{-1} \tag{18}$$

With the average strain concentration tensor evaluated, the abovementioned $\mathrm{E}^{\mathrm{hom}}$ of Eq. (14) can be obtained

$$\mathrm{E}^{\mathrm{hom}} = \mathrm{E}^S : (\mathrm{II} - \varphi_0(\mathrm{II} - \mathrm{S})^{-1}) \tag{19}$$

The Biot tensor arises when anisotropy in porous media is considered and may be obtained by (Dormieux et al., 2006)

$$T^B = \varphi_0 \mathrm{I} : (\mathrm{II} - \mathrm{S})^{-1} \tag{20}$$

On the other hand, if the inhomogeneity is embedded in an isotropic medium and the expression Eq. (16) is valid, a very compact form of fourth-order Eshelby tensor is given as follows

$$\mathrm{S} = \alpha\mathrm{J} + \beta\mathrm{K} \tag{21}$$

being

$$\mathrm{G} = \frac{\alpha}{3k^s}\mathrm{J} + \frac{\beta}{2\mu^s}\mathrm{K}\,;\ \mathrm{E}^S = 3k^s\mathrm{J} + 2\mu^s\mathrm{K}\ ;\ \alpha = \frac{3k^s}{3k^s + 4\mu^s}\ \text{and}\ \beta = \frac{6(k^s + 24\mu^s)}{5(3k^s + 4\mu^s)} \tag{22}$$

Where $\mathrm{J} = \frac{1}{3} I \quad I$; $\mathrm{K} \quad \mathrm{I} \quad \mathrm{J}$, k^s and μ^s are the elastic bulk modulus and the shear modulus of solid phase, respectively.

3.3 The dilute scheme for not interacting prestressed pores.

For a further step pointing to the Biot's problem, a more comprehensive version of the Eshelby's inhomogeneity problem is needed for. It is well known that inclusions in Biot framework are subjected to internal pressure. This fact may be treated as an updated initial condition to the case for empty pores. Dormieux et al. (2006) introduce two new boundary conditions.

$$\Sigma^{SP} = \delta\mathrm{E} : \varepsilon^{SP} + \pi^{SP} \tag{23}$$

$$\varepsilon^{SP} = -\mathrm{G} : \Sigma^{SP} + \Xi \tag{24}$$

In fact, both equations indicate that the inclusion SP embedded in a solid matrix is subjected to a macro strain Ξ ad infinity and also subjected to a constant inner prestress π^{SP} that equals the pore pressure tensor $-p\mathrm{I}$. Solving the previous for ε^{SP}

$$\varepsilon^{SP} \cong \bar{\varepsilon}^{P} = (\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1} : (\Xi - \mathrm{G} : \pi^{SP}) \tag{25}$$

This last expression encompasses the average conditions allowed by the dilute scheme and the possibility of a certain amount of pore pressure $\pi^{SP} = -Ip$. However, and so far, only an isolated inclusion is considered (not interacting). This means a considerable drawback for modelling Biot's problem with multiple interactive inclusions

3.4 The Mori-Tanaka model for different pores condition: re-assessment of the average inclusion strain.

A straightforward extension of the dilute scheme regarding mechanically interacting pores (and volume fraction up to 0.3), may be carried out by means of the Mori-Tanaka average theory (Mori-Tanaka, 1973). Dormieux et al. (2006) provided a bypass by a simple reformulation of the boundary conditions, i.e., fixing the uniform macrostrain boundary condition at infinity in the original Eshelby inhomogeneity problem, to a value Ξ_0 along with a micromacro strain compatibility condition $\bar{\varepsilon} = \Xi$. Additionally, the average strain of the solid phase $\bar{\varepsilon}^{s}$ of the Representative Volume Element (REV) (Lewis and Schrefler, 1998; Anonis et al.) must equals the homogenous environment in which the pore is embedded, "telling" the pore that it is not by itself ($\bar{\varepsilon}^{s}$ shares the environment with average pore strain $\bar{\varepsilon}^{P}$). The inclusion of the magnitude Ξ_0, induce a revision of expression Eq. (25) to meet the updated conditions

$$\varepsilon^{SP} \cong \bar{\varepsilon}^{p} = (\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1} : \Xi_0 \tag{26}$$

$$\bar{\varepsilon}^{s} = \Xi_0 \tag{27}$$

Along with the average microstrain compatibility conditions $\bar{\varepsilon} = \Xi$ and $\Xi = (1-\varphi_0)\bar{\varepsilon}^{s} + \varphi_0\bar{\varepsilon}^{P}$ (Dormieux et al., 2006), the direct relationship between Ξ and Ξ_0 can be written out as follows

$$\Xi = (1-\varphi_0)\Xi_0 + \varphi_0\bar{\varepsilon}^{P} = (1-\varphi_0)\Xi_0 + \varphi_0(\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1} : \Xi_0 = \left((1-\varphi_0)\mathrm{II} + \varphi_0(\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1}\right) : \Xi_0 \tag{28}$$

Solving the previous for Ξ_0 and using $\delta\mathrm{E} = -\mathrm{E}^{S}$ and $\mathrm{S} = \mathrm{G} : \mathrm{E}^{S}$

$$\Xi_0 = \left((1-\varphi_0)\mathrm{II} + \varphi_0(\mathrm{II} - \mathrm{S})^{-1}\right)^{-1} : \Xi \tag{29}$$

These relationships entail a new form for the average strain concentration tensor in the pore space with mechanical interaction, for expression Eq (26) along with Eq (12) leads to

$$\bar{\mathrm{A}}^{P} = (\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1} : \left((1-\varphi_0)\mathrm{II} + \varphi_0(\mathrm{II} + \mathrm{G} : \delta\mathrm{E})^{-1}\right)^{-1} \tag{30}$$

Using the previous expression in the homogenized stiffness tensor expression of eq (14), the so-called Mori-Tanaka version of the drained stiffness tensor is obtained

$$\mathrm{E}^{\mathrm{hom}} = \mathrm{E}^{mt} = \mathrm{E}^{s} : (\mathrm{II} - \varphi_0(\mathrm{II} - \mathrm{S})^{-1} : \left((1-\varphi_0)\mathrm{II} + \varphi_0(\mathrm{II} - \mathrm{S})^{-1}\right)^{-1} \tag{31}$$

In addition, the Biot tensor with pore interaction is (Dormieux et al., 2006; Mroginski et al., 2011)

$$T^{B} = T^{B}\Big|_{MT} = \varphi_0\mathrm{I} : \left(\mathrm{II} - (1-\varphi_0)\mathrm{S}\right)^{-1} \tag{32}$$

In the case of sets (families) of different morphologies between pores (i.e. different orientation, form or content), the above-described framework must be reformulated, including the respective porosities (Dormieux et al 2006). Moreover, one important remark must be made: without loss of generality: it is possible to disregard the dissimilarities between the stress concentration tensors of each phase because no meaningful effect on this tensor due to morphology differences among pores is generally verified. Then, for liquid (l) and gaseous (g) phases

$$\mathbb{A} \quad \bar{\mathbb{A}}^{i} \quad \bar{\mathbb{A}}^{P} \quad \bar{\mathbb{A}}^{l} \quad \bar{\mathbb{A}}^{g} \tag{33}$$

This entails

$$\mathbb{S} \cong \mathbb{S}_l \cong \mathbb{S}_g \text{ through } \bar{\mathbb{A}}^{P} = (\mathbb{II} - \mathbb{S})^{-1} : \left((1-\varphi_0)\mathbb{II} + \varphi_0(\mathbb{II} - \mathbb{S})^{-1}\right)^{-1} \tag{34}$$

Consistently $\mathbb{G} \cong \mathbb{G}_l \cong \mathbb{G}_g$ through $\mathbb{G} = \mathbb{S} : (\mathbb{E}^S)^{-1}$. Also, considering eq (25), bringing back the boundary conditions for interacting pores and adding the fact that we are in the presence of two prestressed inclusions, $-p_l\mathbf{I}$ and $-p_g\mathbf{I}$, taking the place of π^{SP}, the following expressions are valid

$$\bar{\varepsilon}^{l} = (\mathbb{II} - \mathbb{S})^{-1} : (\Xi_0 + p_l \mathbb{G} : \mathrm{I}) \tag{35}$$

$$\bar{\varepsilon}^{g} = (\mathbb{II} - \mathbb{S})^{-1} : (\Xi_0 + p_g \mathbb{G} : \mathrm{I}) \tag{36}$$

$$\bar{\varepsilon}^{s} = \Xi_0 \tag{37}$$

An important remark must be made: For the goals of the present paper, two kinds of pore pressures are considered (liquid and gas), each of which contributes to the mean pressure by the degree of saturation $S_r = \varphi_l / \varphi$ through the following expression

$$\bar{p} = (1 - S_r)p_g + S_r p_l \tag{38}$$

These equations lead to the following relationship $\varphi_g p^g + \varphi_l p^l = \varphi \bar{p}$. With these definitions, the strain average micro- and macrostrain compatibility condition for the partially saturated case is $\Xi = \bar{\varepsilon} = (1-\varphi_0)\bar{\varepsilon}^s + \varphi_l \bar{\varepsilon}^l + \varphi_g \bar{\varepsilon}^g$. Also, the assumed stress concentration condition, $\mathbb{S} \cong \mathbb{S}_l \cong \mathbb{S}_g$ of Eq (34) entails the identity $\varphi_0^i(\mathbb{II} - \mathbb{S}_i)^{-1} = \varphi_0(\mathbb{II} - \mathbb{S})^{-1}$. With all these remarks, the relationship between Ξ_0 and Ξ may be recalculated for partially saturated soils

$$\Xi = (1-\varphi_0)\Xi_0 + \varphi^l \bar{\varepsilon}^l + \varphi^g \bar{\varepsilon}^g = (1-\varphi_0)\Xi_0 + \varphi^l(\mathbb{II} - \mathbb{S})^{-1} : (\Xi_0 + p_l \mathbb{G} : \mathrm{I}) + \varphi^g(\mathbb{II} - \mathbb{S})^{-1} : (\Xi_0 + p_g \mathbb{G} : \mathrm{I}) \tag{39}$$

$$\Xi = (1-\varphi_0)\Xi_0 + \varphi(\mathbb{II} - \mathbb{S})^{-1} : \Xi_0 + \varphi^l(\mathbb{II} - \mathbb{S})^{-1} : p_l \mathbb{G} : \mathrm{I} + \varphi^g(\mathbb{II} - \mathbb{S})^{-1} : p_g \mathbb{G} : \mathrm{I} \tag{40}$$

$$\Xi = (1-\varphi_0)\Xi_0 + \varphi(\mathbb{II} - \mathbb{S})^{-1} : \Xi_0 + \varphi\bar{p}(\mathbb{II} - \mathbb{S})^{-1} : \mathbb{G} : \mathrm{I} \tag{41}$$

Regarding $\varphi \simeq \varphi_0$ and solving for Ξ_0

$$\Xi_0 = \left((1-\varphi_0)\mathbb{II} + \varphi_0(\mathbb{II}-\mathbb{S})^{-1}\right)^{-1} : (\Xi - \varphi_0\overline{p}(\mathbb{II}-\mathbb{S})^{-1} : \mathbb{G} : \mathrm{I}) \quad (42)$$

Replacing Ξ_0 in both average pore strain equations Eq (35) and Eq (36), in a more succinctly form

$$\overline{\varepsilon}^{\alpha} = p_{\alpha}(\mathbb{II}-\mathbb{S})^{-1} : \mathbb{G} : \mathrm{I} + (\mathbb{II}-(1-\varphi_0)\mathbb{S})^{-1} : (\Xi - \varphi_0\overline{p}(\mathbb{II}-\mathbb{S})^{-1} : \mathbb{G} : \mathrm{I}) \quad (43)$$

With $\alpha = l, g$ and being $\overline{\varepsilon}^{\alpha}$ the generic average inclusion (pore) strain

4 ESHELBIAN MECHANICS IN BIOT MICROPOROMECHANICS.

4.1 Configurational forces in Biot's environment

Biot's theory of soil consolidation furnishes the whole environment in which both previous theories, i.e., Eshelbian mechanics and Mori–Tanaka microporomechanics, complement each other, allowing the assessment of a more acute stress–strain field. Then, some of the main concepts hitherto depicted must be reconciled in a comprehensive framework.

Bringing into correspondence the field of stresses due to configurational forces with the field of ordinary stresses that arise in porous media with inclusions, requires a compatible strain field that accounts for both phenomena.

The Eshelbian mechanics for a single pore offer a suitable starting point for tackling the unified stress–strain field problem hinging on the concept of the equivalent strain. It was properly stated in the preceding section that the equivalent strain ε_T^* , is related to macro strains through the expression of Eq (8). Furthermore, in the dilute scheme for microporomechanics, the mean pore strain is expressed in eq (12). Subtracting Ξ from both members both members of the above cited expression follow

$$\overline{\varepsilon}^{p} - \Xi = (\overline{\mathbb{A}}^{P} - \mathbb{II}) : \Xi \quad (44)$$

By integrating the concept of cancelling strain, Eq. (7), the single pore model and the dilute scheme, the following simple yet insightful equation can be proposed

$$e^{C} = \overline{\varepsilon}^{p} - \Xi \quad (45)$$

This, in turn, leads to

$$\varepsilon_T^* = \mathbb{S}^{-1} : (\overline{\varepsilon}^{p} - \Xi) \quad (46)$$

The former rests on the same hypothesis that provides background to the dilute scheme, namely, it is possible to derive the mean average strain in the pore RVE from the strain in Eshelby's inhomogeneity and vice versa. This key relationship paves the way for introducing the equivalent strain concept in the framework of the Mori–Tanaka model, i.e., a simple and direct shortcut between local and average quantities. However, the Mori-Tanaka model involves more than one family of pores, whereas the former involves only one pore family. A direct extension to several families was described in the previous sections, and the same machinery will be enforced here. Denoting by α , a generic family of pores (fluid or gas pore), the equivalent strain for a generic family would be

$$\varepsilon_T^{*\alpha} = \mathbb{S}^{-1} : (\overline{\varepsilon}^{\alpha} - \Xi) \quad (47)$$

Dealing with configurational forces or Eshelby stress is indeed to cope with energy balance. The interaction energy for an inhomogeneous inclusion expressed in Eq (9) (Eshelby, 1951; Alhasadi and Salvatore, 2017) and for the case of several pore families, the additive nature of energy decomposition allows us to consider the energy stored in each family.

More specifically, directing the analysis toward Biot's microporomechanics and denoting by Ω_g the gas domain and Ω_l the liquid domain, the total interaction energy is

$$W^{\text{int}} = -\frac{1}{2}\int_{\Omega g} \Sigma^s : \varepsilon^*_{eq} - \frac{1}{2}\int_{\Omega l} \Sigma^s : \varepsilon^*_{eq} = -\frac{1}{2}\int_{\Omega} \Sigma^s : \quad \varepsilon^{*\,\alpha}_T \tag{48}$$

Provided that the stress concentration tensor is considered the same for all families and, in turn, equal to the average concentration tensor along with a single macro strain, the total equivalent strain may be assessed

$$\varepsilon^*_T = \mathrm{S}^{-1} : (\quad \bar{\varepsilon}^{\alpha} - \Xi) \tag{49}$$

In light of these considerations, the configurational force should be modified to encompass the previously pursued general case (interactive, prestress, and multifamily of pores) by simply substituting ε^*_T from Eq. (49) into Eq. (10)

$$F^{con} \quad \frac{1}{2}\int_D (\mathrm{S}^{-1} : (\quad \bar{\varepsilon}^{\alpha} \quad) : \quad {}^s \tag{50}$$

Moreover, in case of a single family of non-prestressed pores and null porosity, the expression of Eq (43) becomes

$$\bar{\varepsilon}^{\alpha} = (\mathrm{II} - \mathrm{S})^{-1} : (\Xi) \tag{51}$$

Substituting the previous in expression of Eq (50) and after some algebra, it is obtained

$$F^{con} \quad \frac{1}{2}\int_D (\mathrm{S}^{-1} : (\mathrm{A} \quad \mathrm{II}) : \quad) : \quad {}^s \tag{52}$$

Which is equivalent to the expression of Eq (10). Furthermore, let's assume the case in which $\delta\mathrm{E} = \mathrm{E}^{SP} \quad \mathrm{E}^{S} \quad 0$ namely, there are no inclusions (only one substance for both pores and matrix vanishing all inhomogeneity). Then, any version of the concentration tensor shows that $\mathrm{A} \approx \mathrm{II}$. This situation, in turns, compels the configurational forces to a null value

$$F^{con} \quad \frac{1}{2}\int_D (\mathrm{S}^{-1} : (\underbrace{\mathrm{A} \quad \mathrm{II}}_{0}) : \quad) : \quad {}^s \tag{53}$$

This is an obvious consequence and, at the same time, a consistency proof for no determinant configurational force whatsoever should be noted in a homogeneous continuous.

4.2 Eshelby stress and energy involved in the configurational- micro-geomechanical

Following the approach proposed by Alhasadi & Salvatore (2017), an alternative strategy for determining the configurational forces can be pursued using the Eshelby stress tensor. Furthermore, the calculation of these forces necessitates the assessment of the energy involved in the configurational-micro-geomechanical phenomenon. When defects are taken into account, and following the central concepts that lead to the formulation of Eq. (50), a suitable starting point is provided by the following expression

$$b^{B+C} = b^B + \underbrace{\frac{1}{2}(\quad {}^s : \varepsilon^*_T)\,\mathrm{I}_R \quad \frac{1}{2} \quad {}^s \quad \varepsilon^*_T}_{b^C} \tag{54}$$

In the previous, b^{B+C} stands for the Eshelby tensor, b^{B} the Eshelby stress corresponding to an unperturbed stress field and b^{C}, the Eshelby stress due to configurational effects. The expression in Eq. (54) arises from the solution of the integral of the divergence of the Eshelby stress (as represented in Eq. (1) along with Eq. (10)) and the symmetry of the stress tensor. Revisiting the analysis within the configurational-micro-geomechanical framework and considering Eq. (54), it is evident that the energy associated with the current scenario comprises two components: (1) the portion related to the micro-geomechanical aspect, denoted as b^{B}, and (2) the component associated with the configurational aspect of the micro-geomechanical problem, denoted as b^{C}.

$$b^{B+C}=b^{B}+b^{C} \tag{55}$$

$$W^{B+C}=W^{B}+W^{C}=W^{(1)}+W^{(2)} \tag{56}$$

The energy from source (1) must be evaluated from the available energy of the porous media solid phase far from the inclusion. Namely, $W^{(1)}$ is the energy in the soil matrix due to the effective stress solely. Then

$$W^{(1)}=\frac{1}{2}\Xi:\mathrm{E}^{\mathrm{hom}}:\Xi \tag{57}$$

The energy from source (2) derives directly from expression $\frac{1}{2}\Sigma^{s}:\varepsilon_{T}^{*}$ though using equation (49) and recalling the fact that Σ^{s} is the stress far from the inclusion, which, in the context of Biot's theory, is the effective stress

$$W^{(2)}=\frac{1}{2}\Sigma^{s}:(\mathrm{S}^{-1}:(\quad\bar{\varepsilon}^{\alpha}-\Xi))=\frac{1}{2}\Xi:\mathrm{E}^{\mathrm{hom}}:(\mathrm{S}^{-1}:(\quad\bar{\varepsilon}^{\alpha}-\Xi)) \tag{58}$$

Both, $W^{(1)}+W^{(2)}$, represent the total energy available in the whole process. It is a noteworthy feature that, if no pore pressure is considered, i.e., single non prestressed pore, Eq. (57) remains inalterable and Eq. (58) becomes

$$W^{(1)}=\frac{1}{2}\Xi:\mathrm{E}^{\mathrm{hom}}:\Xi \text{ and } W^{(2)}=\frac{1}{2}\Xi:\mathrm{E}^{\mathrm{hom}}:(\mathrm{S}^{-1}:(\mathrm{A}-\mathrm{II}):\Xi) \tag{59}$$

Clearly, $W^{(1)}$ stands for the energy stored in an inhomogeneous solid and $W^{(2)}$ boils down to the integral form of Eq. (9) along with Eq (8). Furthermore, if an homogeneous media condition is added, i.e., $\delta\mathrm{E}=\mathrm{E}^{SP}\quad\mathrm{E}^{S}\quad 0$ and $\mathrm{A}=\mathrm{II}$; then $W^{(1)}$ becomes the standard energy for a solid homogeneous problem and, consequently, $W^{(2)}$ approaches zero along with the configurational forces.

With all the precedent conditions, Eqs. (4), (50) and (51), the divergence of Eq. (57) and the energy equations (57)-(58); stand for the general system of Biot's equations for the unbalanced conservation principle or pseudo momentum within the frame of Eshelbian micromechanics.

5 NUMERICAL SOLUTIONS.

The boundary value problem for coupled consolidation of porous media was extensively studied. In this work we follow the well-known model proposed by Lewis & Schrefler (1998), addressed though the classical Galerkin method.

In its original formulation at most first order derivatives of the displacement and pore pressure fields appear in the governing equations. Therefore, displacement and pressure field discretization require C_0-continuous shape functions that are indicated as $\boldsymbol{N}_u$ and $\boldsymbol{N}_p$, respectively. Then, the FE approximations can be expressed as

$$u=\boldsymbol{N}_u\bar{u};\;\; p=\boldsymbol{N}_p\bar{p} \tag{60}$$

Considering the previous, the following discrete differential equation system is obtained

$$\begin{bmatrix} \boldsymbol{K}_{ss} & \boldsymbol{C}_{sw} \\ \boldsymbol{C}_{ws} & \boldsymbol{K}_{ww} \end{bmatrix} \begin{bmatrix} \dot{\bar{\boldsymbol{u}}} \\ \dot{\bar{p}}_w \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{H}_{ww} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{u}} \\ \bar{p}_w \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{f}}^u \\ \boldsymbol{f}^w \end{bmatrix} \tag{61}$$

Being the matrix expressions

$$\begin{aligned} &\boldsymbol{K}_{ss} = \int_\Omega \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} d\Omega \quad \boldsymbol{H}_{ww} = \int_\Omega (\nabla \boldsymbol{N}_p)^T \frac{\mathbf{k} k^{rw}}{\mu^w} \nabla \boldsymbol{N}_p d\Omega \quad \boldsymbol{P}_{ww} = \int_\Omega \boldsymbol{N}_p{}^T \alpha_{22} \boldsymbol{N}_p d\Omega \\ &\boldsymbol{C}_{sw} = \int_\Omega \boldsymbol{B}^T \alpha_{21} \boldsymbol{N}_p d\Omega \quad \boldsymbol{f}^u = \int_\Omega \boldsymbol{N}_u{}^T \alpha_1 \boldsymbol{g}\, d\Omega + \int_{\Gamma_q^u} \boldsymbol{N}_u^T \alpha_1 \bar{\mathbf{t}}\, d\Gamma_q^u \quad \boldsymbol{f}^w = \int_\Omega (\nabla \boldsymbol{N}_p)^T \mathbf{k}^w \boldsymbol{g}\, d\Omega - \int_{\Gamma_q^w} \boldsymbol{N}_p^T \frac{\mathbf{q}^w}{\rho^w} d\Gamma_q^w \end{aligned} \tag{62}$$

And the Biot's coefficients

$$\alpha_{12} = \alpha_{21} = \alpha S_w \boldsymbol{m}^T, \; \alpha_{22} = \frac{\alpha - n}{K_s} S_w (S_w - p^w \frac{C_w}{n}) + \frac{n S_w}{K_w} - C_w, \; \alpha_1 = 1 - n\rho^s + n S_w \rho^w \quad \mathbf{k}^w = \frac{\mathbf{k} k^{rw}}{\mu^w} \rho^w \tag{63}$$

In the present formulation, a 2D problem is addressed using an 8-node isoparametric quadrilateral finite element (FE). This element has been extensively tested in various scenarios involving multiphase fluid flow in porous media. The Babuska-Brezzi condition is satisfactorily fulfilled by using shape functions for the displacement field, $\boldsymbol{N}_u$ that are of a higher order than those used for the pressure field, $\boldsymbol{N}_p$. Once the boundary value problem of Eq. 62 is solved, the values of $\bar{u}$ and $\bar{p}_w$ are known and also its gradient in each Gauss point. Therefore, the energy $W^{(2)}$ as well as the Eshelby tensor b^C can be evaluated. According to Maugin (2010), among others, the nodal configurational force can be obtained as follows

$$F^{con} = \int_{\Omega_e} \boldsymbol{N}_1 f^{con} d\Omega_e = -\int_{\Omega_e} b^c \nabla \boldsymbol{N}_1 d\Omega_e \tag{64}$$

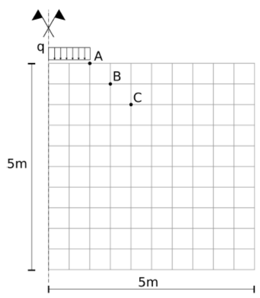


Figure 3 *Strip footing of a saturated granular material, coarsest mesh of 10x10 FE*

Then, the contribution from each element at a particular node I should be assembled to give the discrete value of configurational force at each node. To illustrate the existence, time evolution, and dependency on the adopted spatial discretization, a schematic representation of a real-world foundation problem in the northeastern region of Argentina is presented (see Fig. 3). The problem involves a strip footing on saturated granular material subjected to a uniform load of $q = 100kN / m$, with a width of $B = 5m$ and a depth of $H = 5m$ (only a half of the specimen is discretized due to geometric symmetry). The material properties are assumed as follows: Young's modulus $\mathrm{E}^S = 10.000kPa$, Poisson's ratio $\nu = 0.3$, initial void ratio $e_0 = 0.9$, grain compressibility $K_s = 10 * 10^6 kPa$, and permeability coefficient $k = 8.64 * 10^{-3} m / day$. The boundary conditions are also shown in Fig. 3, along with the coarse finite element (FE) mesh used for the analysis. To analyze the time dependence of configurational forces in a porous media consolidation problem, the evolution of the F_y^{con}-component for nodes A, B, and C (see Fig. 3) is plotted in Fig. 4. Additionally, Fig. 4 shows that the ratio between the configurational forces and mesh size is preserved only for node C, explained by the influence of the free surface and boundary conditions near to nodes A and B

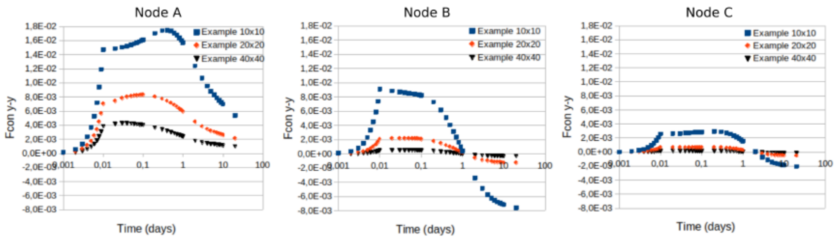


Figure 4 *Time evolution of F_y^{con}-component, corresponding to nodes A, B and C*

Continuing with the same numerical example, in order to provide a qualitative assessment of the magnitude of configurational forces, three different FE meshes of bilinear isoparametric quadrilateral elements (10x10, 20x20, and 40x40) are considered. Snapshots of the F^{con}-vectors at four-time steps are presented in Fig. 5 for each FE mesh employed. Finally, the influence of the material properties of the soil on the vertical component of the configurational forces is analyzed. To this end, the stiffness modulus of a broad range of soils can be approximated using empirical functions proposed by Biarez and Hicher (1994), among others. Those function can be unified in the following form

$$G = A.f(e).OCR^k.(\frac{p}{p_{ref}})^m \text{ (MPa)} \tag{65}$$

Where G is the shear modulus (for small-strain) in MPa, p is the mean effective stress in kPa, p_{ref} is the reference pressure (atmospheric pressure, 100 kPa), OCR is the over-consolidation ratio and $A,\ f(e), k, m$ are correlated functions and parameters which are given in Table 1 for different types of soils

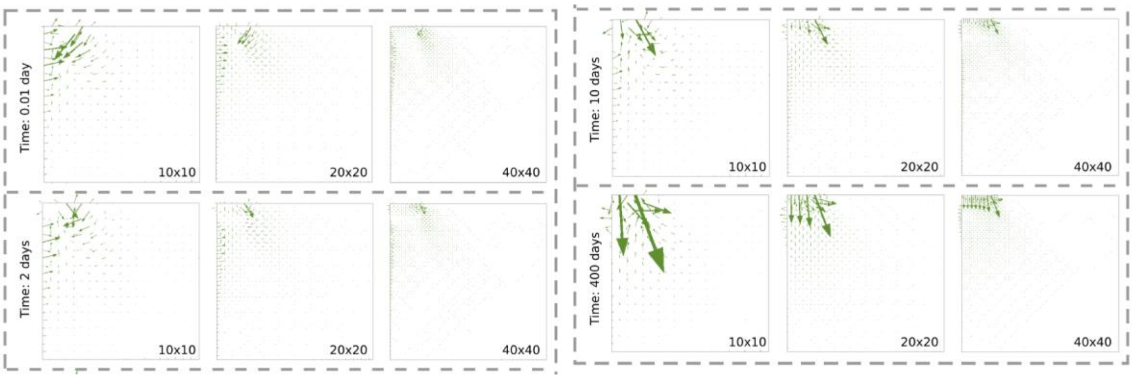


Figure 5 *snapshot at 0.1 days, 80 days and 400 days*

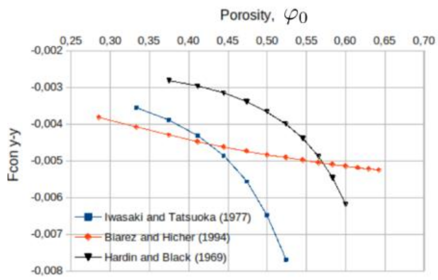


Figure 6 *F_y^{con}-component, for node C, 10x10 mesh, considering three soils typologies*

Soil tested	e_{min}	e_{max}	A	$f(e)$	m	Ref.
Clean uniform sands with $C_u < 1.8$	0.5	1.1	57	$\frac{(2.17-e)^2}{1+e}$	0.4	[54]
All soils with $w_L < 50\%$	0.4	1.8	110	$1/e$	0.5	[55]
Undisturbed crushed sands	0.6	1.5	33	$\frac{(2.97-e)^2}{1+e}$	0.5	[56]

Table 1 *Parameters for estimation of* G *in granular soils using Eq*. 65

Then, according to the approximation Eq. (65) and a given void ratio e, the initial porosity φ_0 it is also known, and the mean stiffness modulus E^S can be straightforward obtained. Therefore, the relation between the intrinsic porosity and the configurational force, for each characteristic soil, can be established and it is presented in Fig.6. The figure shows a wide dispersion among the different soil types, mainly due to the specific nature of each soil.

Since Eq. (65) is an explicit function of the mean effective stress p, it may be of interest to study the influence of this condition on the evolution of the configurational force. For this purpose, the approximation by Biarez and Hicher (1994) is adopted, as it is the equation that best fits the largest number of soils and has the widest range for the void ratio. Then, considering the coarse mesh, the vertical component of the configurational force at Node C is presented in Fig. 7, adopting different values of the mean effective stress p. This figure enables the determination of the configurational force for a characteristic soil type by knowing the porosity and mean effective stress

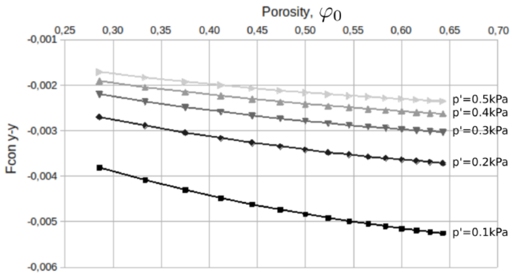


Figure 7 F_y^{con} *-component in node C, 10x10 mesh, assuming Biarez and Hicher (1994) and different values of* p

6 CONCLUSIONS.

A nonlinear mathematical framework for modeling the field equations for an elastic isotropic non saturated soil with pores filled with air and water in the material frame of reference was developed, and the configurational forces were assessed via Eshelbian mechanics giving a radically different role to pore phase

The equivalent strain concept developed for a single pore in references, which is crucial in configurational force analysis, has been extended to multiple families of interacting prestressed pores using micromechanical theory. The present work generalizes the aforementioned references; specifically, through the convergence analysis presented showing that Alhasadi & Salvatore (2017) represent a special case within the framework of the current study

Based upon the interaction energy concept, a simple expression for the energy involved in pinpointing the configurational forces acting in partially saturated soil problems was obtained extending the concept beyond references. Furthermore, the conditions in which this energy boils down to that of references was settled down

Using microporomechanical techniques for strain energy evaluation regarding Bishop's effective stress concept as well as the extended interaction energy concept, a concise expression for the Eshelby/Maxwell second-order stress tensor in the context of pseudomomentum balance was obtained.

Each pore can be considered a defect from the configurational perspective, as it perturbs the overall stress field due to its distinct mechanical properties. This perturbation, in turn, gives rise to configurational forces. The evolution of these forces across different loading stages was analyzed using the Finite Element Method (FEM) along with an in-depth discussion about the problem sensitive with the spatial discretization, material properties, soil characteristics and confinement effective pressure

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