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Coupling Modal Analysis with the BEM for the Transient Response of Bar Structures Interacting with Three-Dimensional Soil Profiles

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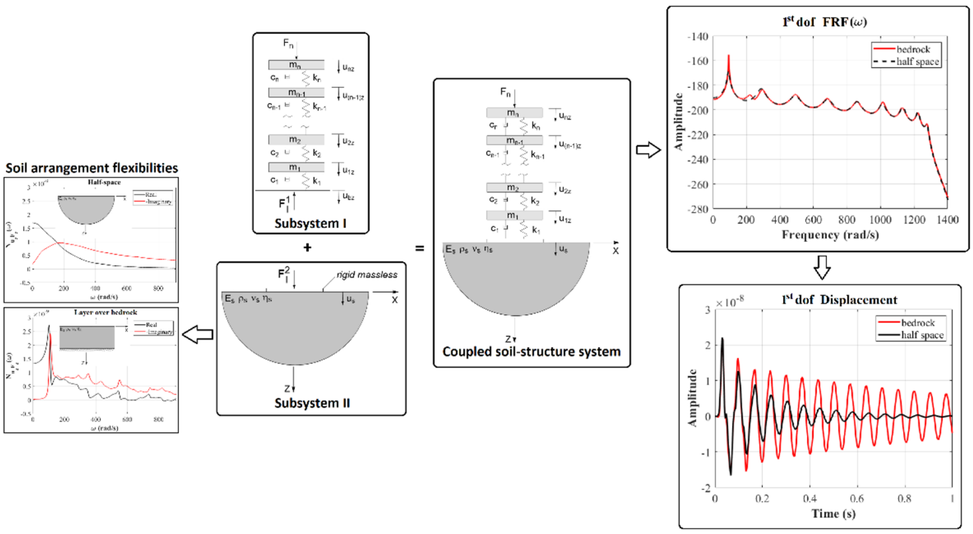
Abstract
This work investigates the transient response of bar structures interacting with three-dimensional soil profiles. The structures are modeled by the Finite Element Method (FEM) and the soil models are described by a three-dimensional Boundary Element Formulation (BEM) in the frequency domain. A classic modal analysis is performed on the structure in terms of the relative displacements with respect to the soil. The dynamic response of the structure is coupled to the soil response, aiming to obtain frequency response functions (FRFs) of the soil-structure system. A new set of modal parameters are extracted from the FRFs of the coupled system. These new parameters allow for the synthesis of a set of orthogonal differential equations in the time domain. These equations are integrated by a classical numerical scheme resulting in the transient response of the structure interacting with the supporting soil. It is shown that for soil profiles that present eigenfrequencies, the system modal basis must be expanded to properly include the soil dynamics. The cases of a structure interacting with a homogeneous half-space and with a horizontal layer over a rigid stratum are considered. The results presented for both soil models are consistent.

Keywords
Dynamic Soil-Structure Interaction, Transient Response, Boundary Element Method, Modal Analysis

Graphical Abstract



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1 INTRODUCTION

The dynamic response of structures interacting with the supporting soil profiles, called Dynamic Soil-Structure Interaction (DSSI), has received the attention of many researchers throughout the last four decades and still is a topic of continuous research. The most significant characteristic of DSSI problems is related to the unbounded dimensions that the soil profile presents. The existence of, at least, one unbounded dimension introduces an effect known as geometric or radiation damping, which is related to the energy that is withdrawn from the structure-foundation system in the form of outgoing and non-reflected waves. So, any attempt to model the dynamics of soils or unbounded domains by numerical methods must be able to take into account the radiation damping, or the Sommerfeld radiation condition. Typical domain-type numerical methods, such as the Finite Element Method (FEM) of the Finite Difference Method (FDM) present finite meshes and are not able to, automatically, consider the geometric damping. The FEM has established itself as the most versatile and used numerical simulation method in solid mechanics. So many attempts have been made to include radiation damping into FEM schemes presenting finite meshes. Examples of these strategies are in the inclusion of “Infinite Elements”, the development of Dirichlet to Newman (DtN) and of Perfectly Matched Layers (PML) schemes to model the unbounded, wave propagating domain (Mesquita and Pavanello, 2005; Zhang et al., 2019).

In the last decades, the Boundary Element Method (BEM) has established itself as the most efficient and accurate method to model problems of wave propagation in unbounded domains and as such the modeling of the dynamic response of unbounded soil profiles. When formulated with the proper “fundamental solution” the BEM only requires the discretization of the surface of the domain being analyzed, reducing the meshing efforts by one dimension. More important for the present study is that the BEM can automatically take into account the radiation condition when dealing with the dynamics of unbounded domains (Dominguez, 1993).

There is a significant amount of research based on the BEM devoted to describing the dynamic response of structures interacting with soil profiles. In these dynamic analyses, the approaches of frequency-domain (stationary) or time-domain (transient) methodologies also apply. Frequency-domain analysis for DSSI problems is well established. The most efficient scheme to analyze DSSI problems is to model the soil by the BEM and the structure by the FEM (Spyrakosa and Xu, 2003; Mehdizadeh et al., 2021).

Time domain analysis using the BEM had a start with the works of Mansur and Brebbia (1982a, 1982b) and has seen a constant evolution ever since. Some important contributions were made by Schanz and Antes (1997), in which the “Operational Quadrature Method” was used to obtain transient BEM solutions. Yerli et al. (1998) used a coupling of finite and infinite elements within the realm of the FEM to obtain the transient solution of unbounded domains. A direct coupling of the FEM with the BEM in the time domain, with the prescription of kinematic compatibility and equilibrium conditions at the soil-structure interface was presented by von Estorff and Prabuki (1990). The coupling of the FEM-BEM domains by a staggered solution was proposed by Rizos and Wang (2002). A time-domain iterative coupled scheme for BEM-FEM was presented by Soares et al. (2004). More recently, the Perfectly Matched Layer (PML) has been used in the FEM methodology to model the transient analysis of structures interacting with horizontally layered soil profiles (Zhang et al., 2019). All the previously mentioned efforts that describe the transient response of structures interacting with unbounded soil profiles are computationally very expensive, which makes the simulation of complex realistic problems almost unattainable. Moreover, the obtained transient responses tend to grow inaccurate for large time periods of analysis.

The aim of the present article is to report the development of an alternative strategy to synthesize the transient response of structures interacting with soils. It builds upon an idea presented by Wu and Smith (1995) in which a modal analysis of the structure was performed with respect to structure displacements relative to the foundation-soil degrees of freedom. The analysis was performed in the frequency domain. The system excitation was in the form of waves impinging upon the foundation. After the coupling of the structure with the soil-foundation subsystem, a set of frequency response functions (FRFs) could be synthesized for the coupled soil-structure system. In that analysis, it was possible to choose arbitrarily the number of structural eigenfrequencies and eigenmodes that would be used to synthesize the FRFs of the coupled system. This is important because in the dynamic response of buildings, the higher structural modes do not play a significant role in the dynamic response. To obtain a transient response to an earthquake excitation Wu and Smith (1995) did multiply the spectrum of the earthquake excitation with the FRFs of the coupled system and performed an inverse FFT (IFFT) operation. So, in this work the FFT algorithm was used to obtain the transient response and the system excitation was given by prescribed incoming waves. Louzada et al. (2019a, 2019b) extended the work of Wu and Smith (1995) to consider external excitations acting directly on the structure. The work also considered the influence of distinct supporting mechanisms on the structural response. Examples of the supporting models were the fixed base, a linear spring, the soil modeled as a half-space, a layer over a rigid bedrock and also a structure supported by a pile embedded in the soil. The transient responses were obtained by the use of the IFFT on the stationary, frequency, responses.

The work presented by Ferraz (2021a) was also based on the FRFs of the coupled soil-structure system and was also able to consider any arbitrary number of modes to describe the structural response but did not use the FFT algorithm to obtain the transient response. From the FRFs of the coupled system, a new set of modal parameters were extracted to build a novel modal basis. This new modal basis was used to synthesize a new set of orthogonal differential equations of motion in the time domain. The direct integration of this uncoupled set of differential equations rendered the transient response of the structure considering the soil influence for the case of external excitations applied directly to the structure.

The present article enlarges the previously described works by allowing the soil model to be more complex and present, itself, eigenfrequencies. The typical model of an unbounded domain presenting eigenfrequencies and eigenmodes is a horizontal layer supported by a rigid bedrock. Now both subsystems present eigenfrequencies. And it is no longer possible to reproduce the dynamics of the coupled system by only considering the number of eigenfrequencies of the original structural system. An expanded modal basis is required to describe the coupled dynamics of both systems. This article uses this expanded modal basis concept to describe the transient dynamic behavior of structures interacting with layer over bedrock. The parameters of the extended modal basis are extracted from the FRFs of the coupled soil-structure system. The results are compared with those stemming from structure interaction with the homogeneous half-space.

Section 2 describes the statement of the problem, the BEM models used to describe the soil response as well as the structural equations of motion in terms of displacements relative to the foundation degrees of freedom. The soil-structure coupling procedure, the expansion of the modal basis and the methodology used to extract modal parameters from the FRFs are described next. Section 3 is dedicated to present numerical examples and conclusions are presented in section 4.

2 STATEMENT OF THE PROBLEM

The typical problem addressed in the present article is presented in Figure 1a, and consists of a linear structure with N degrees of freedom (DOFs) interacting with a soil profile. The structure is modelled using the Finite Element Method (FEM) considering bar elements. Only the displacement in the vertical direction, u_z, is considered.

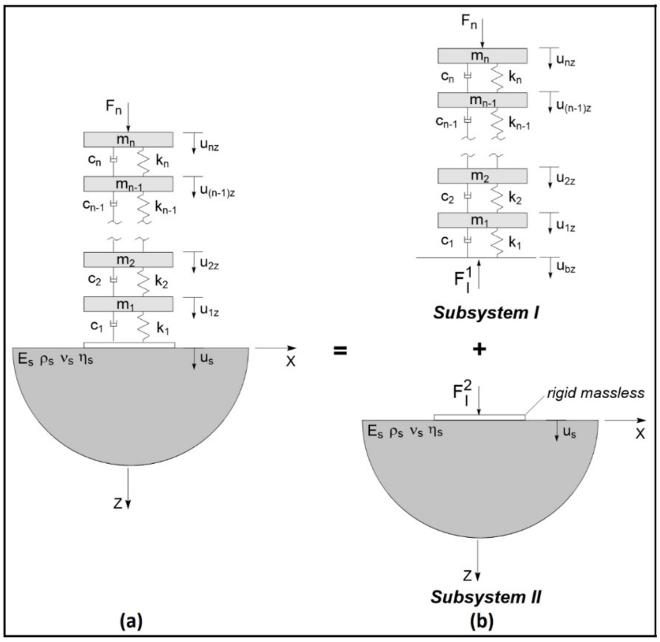

Figure 1 Soil-structure system model and system subdivision.

The soil model depicted in Figure 1a is the classical homogeneous three-dimensional half-space. Nevertheless, in the present work, a horizontal soil layer with thickness, h_S, supported by a rigid bedrock is also analyzed. The three-dimensional soil response in the frequency domain, (ω), is obtained by the Direct Boundary Element Method (DBEM) following the work developed by Carrion et al. (2007). The three-dimensional version of the DBEM allows, in principle, the modeling of soil profiles with arbitrary geometry (Carrion et al., 2007). It is also assumed that the structure interacts with a rigid and massless foundation at the soil-foundation interface. The considered excitation are external forces, F_{ext}, applied to the foundation degrees of freedom.

The system presented in Figure 1a is subdivided into two subsystems, respectively, the structure (Subsystem I) and the soil with the rigid and massless foundation (Subsystem II), as shown in Figure 1b. The interface forces between the structure and the soil-foundation systems are, respectively F_I^1 and F_I^2 . The responses of each subsystem are obtained according to the formulations described in the sequence.

2.1 Soil-foundation formulation

The three-dimensional Direct Boundary Element Method (DBEM) with constant quadrilateral elements is used to derive the soil response in the frequency domain, (ω) . The soil model is characterized by its density, ρ_S , Young modulus, E_S , Poisson ratio, ν_S , and the internal damping factor, η_S . A frequency independent, constant, damping factor, η_S = *cte*, is considered for the soil in the present analysis (Beskos, 1987). The three-dimensional (3D) Boundary Element formulation in the frequency domain is based on the “frequency domain full space fundamental solution” (Dominguez, 1993). The BEM implementation described in (Carrion, 2002) allows for the dynamic response of 3D soils under applied surface tractions and also for the interaction of the soil with rigid surface or embedded foundations. Different soil profiles can be modeled, such as the homogeneous half-space, layer over a horizontal bedrock or layer on a non-horizontal bedrock, among others (Carrion et al., 2007). In the next paragraphs, the formulation of a rigid and massless foundation interacting with a 3D soil layer supported by a rigid bedrock is described. The formulation for rigid foundations interacting with the half-space can be obtained by a simplification of the case shown in this article (Carrion, 2002).

Figure 2 shows a rigid and massless foundation with dimensions $2a \times 2a$ bonded at the surface of a horizontal soil layer with thickness h_s and supported by a rigid bedrock. The boundary at the soil-foundation interface is Γ_f , the soil-free surface is Γ_{s1} and the interface soil-rigid bedrock is Γ_{s2} . At this first step is it convenient to consider that $\Gamma_s = \Gamma_{s1} \cup \Gamma_{s2}$, which represents the soil boundaries that are not interacting with the rigid foundation interface.

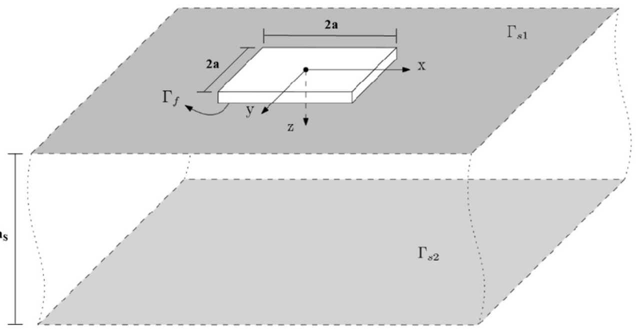

Figure 2 Rigid and massless foundation at the surface of a 3D horizontal layer over bedrock

The Boundary Element equations can be formulated in terms of the surface displacements, U , and surface tractions, T . Assuming that the surface displacements, U , and tractions can also be subdivided according to $\Gamma_S(U_S, T_S)$ and $\Gamma_f(U_f, T_f)$, the discretized BEM equations in matrix form can be written as (Carrion, 2002)

$$\begin{bmatrix} H_{ff} & H_{fs} \\ H_{sf} & H_{ss} \end{bmatrix} \begin{Bmatrix} U_f \\ U_s \end{Bmatrix} = \begin{bmatrix} G_{ff} & G_{fs} \\ G_{sf} & G_{ss} \end{bmatrix} \begin{Bmatrix} T_f \\ T_s \end{Bmatrix} \tag{1}$$

The displacements at the soil-foundation interface, $\{U_f\}$, can be related to the rigid body degrees of freedom of the massless and rigid foundation, $\{U_0\} = \{U_{x0} \quad U_{y0} \quad U_{z0} \quad \Phi_{x0} \quad \Phi_{y0} \quad \Phi_{z0}\}^T$, using a kinematic compatibility matrix, $[C_K]$

$$\{U_f\} = [C_K]\{U_0\} \tag{2}$$

The tractions of the soil-foundation interface, $\{T_f\}$, are related to the vector of the resulting external forces, $\{F\} = \{F_x \quad F_y \quad F_z \quad M_x \quad M_y \quad M_z\}^T$, applied to the rigid and massless foundation through a matrix of equilibrium equations, $[D]$

$$\{F\} = [D]\{T_f\} \tag{3}$$

The problem can be rewritten in terms of the rigid body displacements of the rigid foundation, $\{U_0\}$, and the applied external forces, $\{F\}$ (Carrion, 2002)

$$\begin{bmatrix} H_{ff}C & H_{fs} & -G_{ff} & -G_{fs} \\ H_{sf}C & H_{ss} & -G_{sf} & -G_{ss} \\ 0 & 0 & D & 0 \end{bmatrix} \begin{Bmatrix} U_0 \\ U_s \\ T_f \\ T_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix} \tag{4}$$

In equation (4), if the external forces are known, it is possible to determine the foundation rigid body displacements, U_0, and the remaining soil-surface displacements, U_S, the tractions at the soil-foundation interface, T_f, and the tractions at the remaining soil surface, T_S. Assuming the tractions, $\{T_{s1}\}$, on the free surface, Γ_{s1}, and the displacements, $\{U_{s2}\}$, on the soil bedrock foundation, Γ_{s2}, are known, equation (4) can be rearranged as

$$\begin{bmatrix} H_{ff}C & -G_{ff} & H_{fs1} & -G_{fs2} \\ H_{sf}C & -G_{sf} & H_{ss1} & -G_{ss2} \\ 0 & D & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_0 \\ T_f \\ U_{s1} \\ T_{s2} \end{Bmatrix} = \begin{bmatrix} -H_{fs2} & G_{fs1} \\ -H_{ss2} & G_{ss1} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} U_{s2} \\ T_{s1} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix} \tag{5}$$

The boundary conditions for the tractions on the free surface of the soil, Γ_{s1}, are assumed to be zero, $\{T_{s1}\}=0$. The soil is considered completely bonded at the soil-bedrock surface, Γ_{s2}, and the corresponding boundary condition is $\{U_{s2}\}=0$. Considering these boundary conditions, equation (5) can be written as

$$\begin{bmatrix} H_{ff}C & -G_{ff} & H_{fs1} & -G_{fs2} \\ H_{sf}C & -G_{sf} & H_{ss1} & -G_{ss2} \\ 0 & D & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_0 \\ T_f \\ U_{s1} \\ T_{s2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix} \tag{6}$$

This system of equations (6) can be solved to find the components of the rigid body displacements, $\{U_0\}$, for a given frequency, (ω). To obtain the rigid body displacements, the vector of the external forces, $\{F\}$, assumes a sequence of unit value loads, i.e., $\{F_x\}=\{F_x=1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\}^T$, $\{F_y\}=\{0 \;\; F_y=1 \;\; 0 \;\; 0 \;\; 0 \;\; 0\}^T$ and so on. The columns of the displacements resulting from the 6-unit value external force vectors may be organized in matrix form, to generate a dynamic flexibility (compliance) matrix, $[N(\omega)]$, of the rigid and massless foundation interacting with the soil (Carrion, 2002). This frequency dependent flexibility matrix relates the vector of the external forces applied to the rigid foundation, $\{F\}$, to the rigid body displacements of the foundation, $\{U_0\}$

$$\{U_0\} = \frac{1}{G_S\, a}[N(\omega)]\{F\} \tag{7}$$

In equation (7) a is half the length of the foundation's side and G_S is the shear modulus of the soil. Figures 3a and 3b show the real and imaginary parts of the flexibility functions, $N_{u\,z\,F\,z}(\omega)$, relating the external vertical excitation, F_z, to the vertical rigid foundation degree of freedom, u_z, for the case of a homogeneous half-space (Figure 3a) and a horizontal layer over bedrock (Figure 3b). The layer depth is $h_s = 5a$. These results were obtained for the soil parameters: $a=1m$, $E_S = 234MPa$, $\rho_s = 2700\ kg/m^3$, $\nu_s = 0.3$ and $\eta_s = 0.01$ (Carrion, 2002).

These figures show a large difference between the dynamic flexibility of a rigid and massless foundation interacting with the homogeneous half-space (Figure 3a) and with the horizontal layer over a bedrock (Figure 3b). The fact that the horizontal layer has a finite height, h_s, introduces natural frequencies in the vertical direction. This point will be addressed again in the formulations and results that will follow.

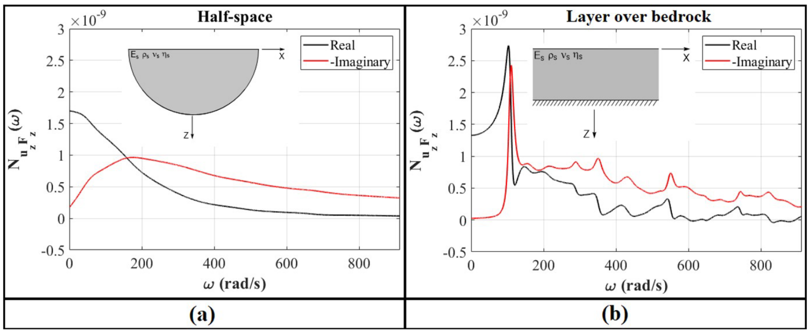

Figure 3 Comparison flexibilities $N_{uzFz}(\omega)$ for the half-space (a) and for a layer over bedrock (b).

2.2 Equations of motion for the Structure in relative coordinates

Consider the vector of the total displacements of the structural system, $\{u_z(t)\} = \{u_{1z}(t) \quad u_{2z}(t) \quad \dots \quad u_{Nz}(t)\}^T$, and the vertical displacement of the rigid foundation, $u_{bz}(t)$. A vector of the structural displacements, $\{u_{relz}\}$, relative to the rigid foundation vertical displacement vertical can be defined as: $\{u_{relz}(t)\} = u_z(t)\{1\} - \{u_{bz}(t)\}$. Using the relative displacements, the time domain equations of motion of the structure with mass matrix, $[M]$, damping and stiffness matrices $[C]$ and $[K]$, respectively, can be written as

$$\begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} \begin{Bmatrix} \ddot{u}_{1z}(t) \\ \ddot{u}_{2z}(t) \\ \vdots \\ \ddot{u}_{nz}(t) \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & \dots & 0 \\ -c_2 & c_2+c_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix} \begin{Bmatrix} \dot{u}_{1z}(t)-\dot{u}_{bz}(t) \\ \dot{u}_{2z}(t)-\dot{u}_{bz}(t) \\ \vdots \\ \dot{u}_{nz}(t)-\dot{u}_{bz}(t) \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & \dots & 0 \\ -k_2 & k_2+k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix} \begin{Bmatrix} u_{1z}(t)-u_{bz}(t) \\ u_{2z}(t)-u_{bz}(t) \\ \vdots \\ u_{nz}(t)-u_{bz}(t) \end{Bmatrix} = \begin{Bmatrix} F_{1z}(t) \\ F_{2z}(t) \\ \vdots \\ F_{nz}(t) \end{Bmatrix} \tag{8}$$

where, $\{F_{ext}(t)\} = \{F_{1z}(t) \quad F_{1z}(t) \quad \cdots \quad F_{nz}(t)\}^T$ is the vector of external forces applied to the rigid and massless foundation. Equation (8) can be rearranged in terms of the relative structural displacement, $\{u_{relz}\}$, to yield (Ferraz, 2021)

$$[M]\{\ddot{u}_{relz}(t)\} + [C]\{\dot{u}_{relz}(t)\} + [K]\{u_{relz}(t)\} = \{F_{ext}(t)\} - [M]\{1\}\ddot{u}_{bz}(t) \tag{9}$$

In order to obtain the stationary, frequency response of Subsystem I, equation (9) must be transformed from the time domain to the frequency domain with the application of the Fourier Transform leading to

$$-\omega^2[M]\{U_{relz}(\omega)\} + i\omega[C]\{U_{relz}(\omega)\} + [K]\{U_{relz}(\omega)\} = \{F_{ext}(\omega)\} + \omega^2[M]\{1\}U_{bz}(\omega) \tag{10}$$

Displacements in the frequency domain, (ω), are described in capital letters. In the present study, the damping matrix is assumed to be proportional to the mass and stiffness matrix, $[C] = \mu[M] + \beta[K]$, with the proportional coefficients μ and β (Caughey, 1960; Chopra, 2012). Next, a classical modal analysis is performed on the structural system described in equation (10). The solution of the linear eigenvalue problem will deliver the structural eigenvalues (natural frequencies), ω_i, (i=1, n), the modal damping coefficients, ξ_i, (i=1, n) and the eigenmodes (vibration modes), $\{\phi_i\}$, (i=1, n) for the original structure in terms of the relative displacements. The eigenvectors can be organized into the modal matrix, $[\Phi]$ (Chopra, 2012; Fu and He, 2001). The modal matrix allows the transformation of the vector with structural physical degrees of freedom expressed in relative coordinates, $\{U_{relz}(\omega)\}$, into the vector of modal coordinates, also in relative coordinates, $\{Q_{relz}(\omega)\}$

$$\{U_{relz}(\omega)\} = [\Phi]\{Q_{relz}(\omega)\} \tag{11}$$

By substituting equation (11) in equation (10) and pre-multiplying the resulting expression by the transposed of the modal matrix, $[\Phi]^T$, leads to

$$[\Phi]^T(-\omega^2[M] + i\omega[C] + [K])[\Phi]\{Q_{relz}(\omega)\} = [\Phi]^T\{F_{ext}(\omega)\} + \omega^2[\Phi]^T[M]\{1\}U_{bz}(\omega) \tag{12}$$

Consider the orthogonal conditions for classical modal analysis of linear systems related to the modal matrix, $[\Phi]$ (Fu and He, 2001)

$$\begin{cases} [\Phi]^T_{n\times n}[M]_{n\times n}[\Phi]_{n\times n} = [I]_{n\times n} \\ [\Phi]^T_{n\times n}[C]_{n\times n}[\Phi]_{n\times n} = \left[{}^{*}2\xi_n\omega_n\ {}_{*}\right]_{n\times n} \\ [\Phi]^T_{n\times n}[K]_{n\times n}[\Phi]_{n\times n} = \left[{}^{*}\omega_n^2\ {}_{*}\right]_{n\times n} \end{cases} \tag{13}$$

In which $[I]$ is the identity matrix, ω_n and ξ_n represent, respectively, the system n-th natural frequency and modal damping factor. Applying the transformations (13) to equations (12) results in the uncoupled set of equations in terms of the modal response of the structure subsystem, in relative coordinates, and in the frequency domain

$$\{Q_{relz}(\omega)\} = [H(\omega)]\Big([\Phi]^T\{F_{ext}(\omega)\} + \{\Gamma(\omega)\}U_{bz}(\omega)\Big) \tag{14}$$

where the transfer function, $[H(\omega)]$, and the generalized modal load coefficient, $\{\Gamma(\omega)\}$, are defined, respectively, by (Ferraz, 2021; Wu and Smith, 1995)

$$[H(\omega)] = (-\omega^2[I] + i\omega[{}^{*}2\xi_n\omega_n\ {}_{*}] + [{}^{*}\omega_n^2\ {}_{*}])^{-1} \tag{15}$$

$$\{\Gamma(\omega)\} = \omega^2[\Phi]^T[M]\{1\} \tag{16}$$

The relative physical displacement of the structure, $\{U_{relz}(\omega)\}$, can be recovered using the equation (11)

$$\{U_{relz}(\omega)\} = [\Phi][H(\omega)]\Big([\Phi]^T\{F_{ext}(\omega)\} + \{\Gamma(\omega)\}U_{bz}(\omega)\Big) \tag{17}$$

Equations (17) express the relative displacement of the structure degrees of freedom, $\{U_{relz}(\omega)\}$, as a function of the external excitation, $\{F_{ext}(\omega)\}$, and the rigid foundation displacement, $U_{bz}(\omega)$. The soil influence is not yet incorporated into the structural response. One important characteristic of this formulation is that it is possible to choose an arbitrary number of structural modes in equation (15) to represent the dynamics of the structure (Ferraz, 2021; Wu and Smith, 1995). For structures with a large number of degrees of freedom and modes, the formulation allows investigating how many structural modes need to be considered to obtain a structural response within a given accuracy.

2.3 Coupling soil and structures responses

The objective of this session is to connect the soil and structural response formulations in the frequency domain, which will lead to the synthesis of Frequency Response Functions (FRFs) of the coupled system.

After defining the dynamic response of the soil, equation (7), and of the structural subsystem, equation (17), the coupling between the subsystems is carried out by the balance of forces and kinematic compatibility at the soil-structure interface, Γ_f. First, a balance of forces in Subsystem I (structure) is performed using the total structural degrees of freedom, $\{u_z(t)\}$, in the time domain

$$-F_{1z}(t) - F_{2z}(t) \cdots - F_{nz}(t) + F_{1z}^I(t) + m_1\ddot{u}_{1z}(t) + m_2\ddot{u}_{2z}(t) \ldots + m_n\ddot{u}_{nz}(t) = 0 \tag{18}$$

In equation (18), $F_I^1(t)$ is the interface force between the rigid foundation and the 1-st structural degree of freedom (see Figure 1b). Rewriting equation (18) in matrix form will lead to

$$F_I^1(t) = \{1\}^T\{F_{ext}(t)\} - \{1\}^T[M]\{u_z(t)\} \tag{19}$$

The equilibrium equation (19) can also be transformed to the frequency domain by the Fourier Transform

$$F_I^1(\omega) = \{1\}^T \{F_{ext}(\omega)\} + \omega^2 \{1\}^T [M] \{U_z(\omega)\} \tag{20}$$

Now consider, in equation (7), only the vertical soil-foundation compliance, $N_{uzFz}(\omega)$ relating the vertical displacement of the rigid and massless foundation interacting with the soil, $U_{bz}(\omega)$ to the vertical foundation-structure interface force, $F_I^2(\omega)$ (see Figure 1b)

$$U_{bz}(\omega) = \frac{1}{G_s\, a} N_{uzFz}(\omega) F_I^2(\omega) \tag{21}$$

The compliance, $N_{uzFz}(\omega)$, can be inverted to produce a vertical dynamic stiffness of the soil-foundation, $K_s(\omega)$, defined by

$$K_s(\omega) = G_s\, a \left[N_{uzFz}(\omega) \right]^{-1} \tag{22}$$

Considering equations (21) and (22), the displacement of the rigid and massless foundation interacting with the soil in the frequency domain, $U_{bz}(\omega)$, can be related to the external force applied to the foundation-structure interface, $F_I^2(\omega)$, through the dynamic vertical stiffness, $K_s(\omega)$

$$F_I^2(\omega) = K_s(\omega)\, U_{bz}(\omega) \tag{23}$$

Equilibrium conditions at the rigid foundation-structure interface prescribe that

$$F_I^1(\omega) + F_I^2(\omega) = 0 \tag{24}$$

Equations (20), (23) and (24) can be rearranged resulting in

$$\{1\}^T \{F_{ext}(\omega)\} + \omega^2 \{1\}^T [M] \{U_z(\omega)\} = K_s(\omega) U_{bz}(\omega) \tag{25}$$

On the other hand, the total response of the structure degrees of freedom, $\{U_z(\omega)\}$, is given by the sum of the relative displacements, $\{U_{relz}(\omega)\}$, and the displacement from the soil, $U_{bz}(\omega)$

$$\{U_z(\omega)\} = \{U_{relz}(\omega)\} + \{1\} U_{bz}(\omega) \tag{26}$$

Reorganizing equations (17), (25) and (26) will result in

$$K_s(\omega) U_{bz}(\omega) = \{1\}^T \{F_{ext}(\omega)\} + \omega^2 \{1\}^T [M] \Big([\Phi][H(\omega)] \Big([\Phi]^T \{F_{ext}(\omega)\} + \{\Gamma(\omega)\} U_{bz}(\omega) \Big) + \{1\} U_{bz}(\omega) \Big) \tag{27}$$

Collecting the terms $U_{bz}(\omega)$ and $\{F_{ext}(\omega)\}$ in equation (27) leads to

$$U_{bz}^{dssi}(\omega) = \{S_{bz}(\omega)\} \{F_{ext}(\omega)\} \tag{28}$$

where,

$$\{S_{bz}(\omega)\} = \frac{\{1\}^T + \{\Gamma(\omega)\}^T [H(\omega)][\Phi]^T}{K_s(\omega) - \omega^2 \{1\}^T [M] \{1\} - \{\Gamma(\omega)\}^T [H(\omega)] \{\Gamma(\omega)\}} \tag{29}$$

Equation (28) furnishes the vertical displacement of the rigid and massless foundation interacting with the soil, $U_{bz}^{dssi}(\omega)$, considering the effects of the soil dynamics, $K_s(\omega)$, and the dynamics of the structure described in terms of modal quantities, $[H(\omega)]$, as can be seen by analyzing equation (29). The upper index 'dssi' has been added to the displacement to indicate that it already contains the soil-structure effect. The excitation remains the external forces applied directly to the n foundation degrees of freedom, $\{F_{ext}(\omega)\}$.

To determine the total structure response with soil influence, $\{U_z^{dssi}(\omega)\}$, the soil-rigid foundation displacement, $U_{bz}^{dssi}(\omega)$, determined by expression (28) must be added to the vector of the structure relative displacements, $\{U_{relz}(\omega)\}$, given by equation (17)

$$\{U_z^{dssi}(\omega)\} = \{S_{est}(\omega)\}\{F_{ext}(\omega)\} \tag{30}$$

where,

$$[S_{est}(\omega)] = [\Phi][H(\omega)]\left([\Phi]^T + \{\Gamma(\omega)\}\{S_{bz}(\omega)\}\right) + \{1\}\{S_{bz}(\omega)\} \tag{31}$$

The matrix $[S_{est}]$ represents a dynamic compliance matrix relating the total displacement of the structure degrees of freedom, $\{U_z^{dssi}(\omega)\}$, under an external excitation, $\{F_{ext}(\omega)\}$, applied at the structure degrees of freedom and already considering the soil effects, $K_s(\omega)$, for an arbitrary number of modes to describe the structure dynamics, $[H(\omega)]$.

Equation (30) allows the construction of modified frequency response functions (FRFs) for the structure, using an arbitrary number of structural modes, and already considering the soil-structure interaction effects. As will be explained in the next sessions, structural modal parameters may be extracted from these modified FRFs and used to reconstruct a system of orthogonal equations of motion in the time domain in terms of modal parameters.

2.4 Modal parameters extraction from modified FRFs

The Rational Fraction Polynomial Method (RFPM) was chosen to extract the modal parameters (natural frequencies, damping factors and modal forms) from the FRFs containing soil-structure interaction soil effects given in equation (30). The RFPM is a classic extraction method presented in Ewins (2000), and a more detailed formulation can be seen in Richardson and Formenti (1982) and Ferraz (2021). This method is based on the approximation of the FRF curve by means of complex orthogonal polynomials, represented in equation (32). From this approximation, the relationships between the poles and residuals of the partial fractions and the modal quantities are determined.

$$H(\omega) = \frac{\sum_{k=0}^{2N-1} a_k (i\omega)^k}{\sum_{k=0}^{2N} b_k (i\omega)^k} \tag{32}$$

In equation (32) N represents the number of modes in the FRF to be adjusted. The modal quantities extracted from the formulation (32) are eigenfrequencies including the soil-foundation effects, ω_i^{dssi}, as well as the modal damping coefficients, ξ_i^{dssi}, and eigenvectors, $\{\phi_i^{dssi}\}$. Results of the extracted modified modal quantities using this methodology will be presented in section 3.

2.5 Transient response by modal superposition

Starting from the modal quantities, represented by the natural frequencies, ω_i^{dssi}, the modal damping factors, ξ_i^{dssi}, of the i-th mode and the matrix of modal shapes, $[\Phi^{dssi}]$, a set of orthogonal, uncoupled, equations of motion in the time domain, in terms of the modal coordinates, $\{q_z^{dssi}(t)\}$, may be constructed to represent the dynamic response of the system under analysis (Fu and He, 2001; Ferraz, 2021)

$$[I]\{\ddot{q}_z^{dssi}(t)\} + [2\xi_n^{dssi}\,\omega_n^{dssi}]\{\dot{q}_z^{dssi}(t)\} + [(\omega_n^{dssi})^2]\{q_z^{dssi}(t)\} = [\Phi^{dssi}]^T\{F_{ext}(t)\} \tag{33}$$

Equation (33) represents a set of uncoupled ordinary differential equations that can be directly integrated in time domain, delivering the response in terms of the modal coordinates including the soil-structure effects, $\{q_z^{dssi}(t)\}$. The displacement response in physical coordinates, $\{u_z^{dssi}(t)\}$, can be obtained through equation (11)

$$\{u_z^{dssi}(t)\} = [\Phi^{dssi}]\{q_z^{dssi}(t)\} \tag{34}$$

In the present work, the Newmark method (Chopra, 2012) was chosen to perform the numerical integration of the equations of motion (33), due to its unconditional stability for any time step when the parameters γ = 0.25 and β = 0.5 are used (Chopra, 2012).

2.6 Expanded modal basis

In soil models with unbounded dimensions, like the half-space, there are no natural frequencies or normal vibration modes. On the other hand, for soil profiles with a limited dimension, such as a layer of finite depth over a rigid stratum or bedrock, there are natural frequencies and normal vibration modes along the finite dimension. For soil profiles that present eigenfrequencies, it might be necessary to expand the original modal basis of the structure to include the new eigenfrequencies that may appear on the FRFs of the coupled soil-structure system due to soil response.

This influence can be seen in the example of Figure 4, which simulates a soil-structure system in which the soil is represented by a mass-spring-damper system with 2 dofs and the structure with 3 dofs. Note that the FRFs of each separate subsystem only show resonances referring to their number of dofs. However, when the coupling is performed, 5 resonances can be observed in the FRF of the coupled system.

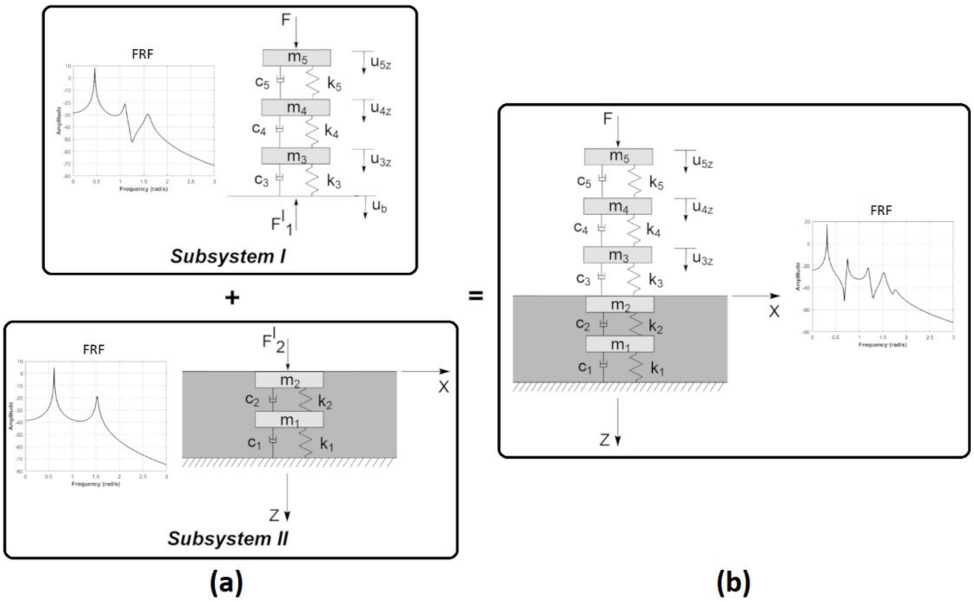

Figure 4 Soil influence on structure FRFs.

So, frequency response functions obtained by equations (30) may present more resonances than the structure's N original degrees of freedom, depending on the characteristics of the soil profile supporting the foundation and structure. For these cases, it is necessary to expand the modal basis to include a larger number of system degrees of freedom S, which is the sum of the original N structural dofs and the new degrees of freedom added to the system by the inclusion of the soil response. The equation transforming the expanded modal displacements, $\{q_z^{dssi}(t)\}_{s\times 1}$, with S degrees of freedom is given by

$$\{u_z^{dssi}(t)\}_{n\times 1} = [\Phi^{dssi}]_{n\times s}\{q_z^{dssi}(t)\}_{s\times 1} \tag{35}$$

The expanded modal basis is given by $[\Phi^{dssi}]_{n\times s}$. The orthogonality conditions (13) must also be changed to reflect the expansion of the modal basis

$$\begin{cases} \left[\Phi^{dssi}\right]_{n\times s}^{T}\left[M\right]_{n\times n}\left[\Phi^{dssi}\right]_{n\times s}=\left[I\right]_{s\times s} \\ \left[\Phi^{dssi}\right]_{n\times s}^{T}\left[C\right]_{n\times n}\left[\Phi^{dssi}\right]_{n\times s}=\left[{}^{*}2\xi_{s}\,\omega_{s}\,{}_{*}\right]_{s\times s} \\ \left[\Phi^{dssi}\right]_{n\times s}^{T}\left[K\right]_{n\times n}\left[\Phi^{dssi}\right]_{n\times s}=\left[{}^{*}(\omega_{s})^{2}\,{}_{*}\right]_{s\times s} \end{cases} \tag{36}$$

It should be noted that this expanded system has now s eigenfrequencies and s eigenmodes. Based on equations (36) an updated set of orthogonal equations of motion in the time domain can be synthesized

$$[I]_{s\times s}\{\ddot{q}_z^{dssi}(t)\}_{s\times 1}+[{}^{*}2\xi_s^{dssi}\,\omega_s^{dssi}{}_{*}]_{s\times s}\{\dot{q}_z^{dssi}(t)\}_{s\times 1}+[{}^{*}(\omega_s^{dssi})^2{}_{*}]_{s\times s}\{q_z^{dssi}(t)\}_{s\times 1}=[\Phi^{dssi}]_{s\times n}^{T}\{F_{ext}(t)\}_{n\times 1} \tag{37}$$

The formulation resulting in equation (37) indicates that a new, expanded, system of equations of motion with s degrees of freedom is synthesized to cope with the modifications in the time domain structural response to include the soil effect.

3 RESULTS

3.1 Systems and input data

For the numerical analysis, two distinct models are investigated. These are depicted in Figures 5a and 5b. The structure is composed of bar elements with 9 dofs and is subjected to an excitation force applied to the 9-th dof. The soil models are the homogeneous half-space (Figure 5a) and a horizontal layer of depth, $h_s = 5a$, with a being the foundation half-width (Figure 5b). The soil properties are given in Table 1. The soil dynamic flexibility, $N_{uzFz}(\omega)$, for System I and II is shown, respectively in Figures (3a) and (3b). Table 2 presents the properties of the structure, including the mass and stiffness values for each element, along with the coefficients of proportional damping μ and β.

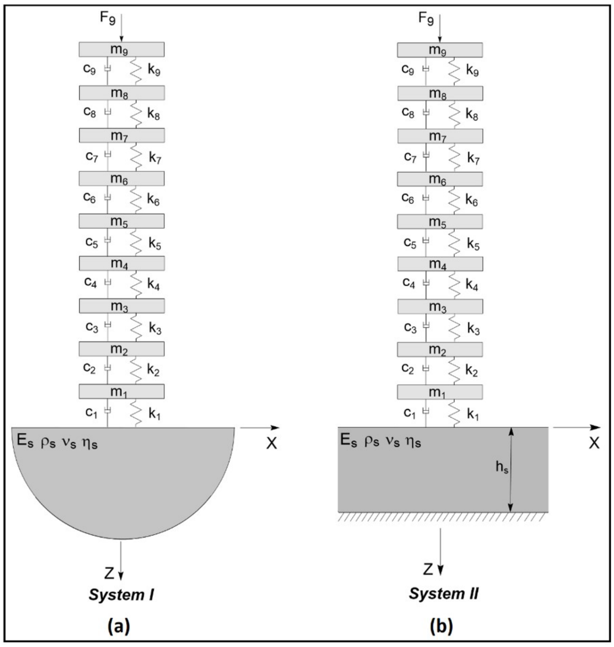

Figure 5 Soil-structure systems with different soil profiles.

Table 1 Soil properties.

Homogeneous half-space and layer over bedrock	
Density (kg/m³)	ρ_s=2700
Young Modulus (MPa)	E_s=234
Shear Modulus (MPa)	G_s=90
Shear Velocity (m/s)	v_s=341.6
Poisson's ratio	Y=0.3
Damping	η=0.01

Table 2 Structure properties.

Bar element	
mass (kg)	m_i=20358
Stiffness (MPa)	k_i=101790
Damping coefficient	μ=2.0656
	α=6.5581*10^{-6}

3.2 Structure modal data without soil influence (fixed base)

For comparison purposes, the classic modal analysis was carried out only for the structure, considering a fixed base, obtaining the natural frequencies and damping factors for the 9 modes, as shown in Table 3.

Table 3 Modal parameters of the structure (fixed base).

Natural frequency (rad/s)	Damping Factor
ω_1=107.0	ξ_1=0.0100
ω_2=318.2	ξ_2=0.0043
ω_3=520.7	ξ_3=0.0037
ω_4=708.9	ξ_4=0.0038
ω_5=877.9	ξ_5=0.0041
ω_6=1022.8	ξ_6=0.0044
ω_7=1139.9	ξ_7=0.0046
ω_8=1225.9	ξ_8=0.0049
ω_9=1278.5	ξ_9=0.0050

3.3 Modified FRFs for the structure interacting with the soil models

Using the methodology summarized in equation (30), a set of 9 FRFs for the structure interacting with both soil profiles is shown in Figure 6. The black dashed line corresponds to the case of the homogeneous half-space and the red line correspond to the layer over a rigid stratum.

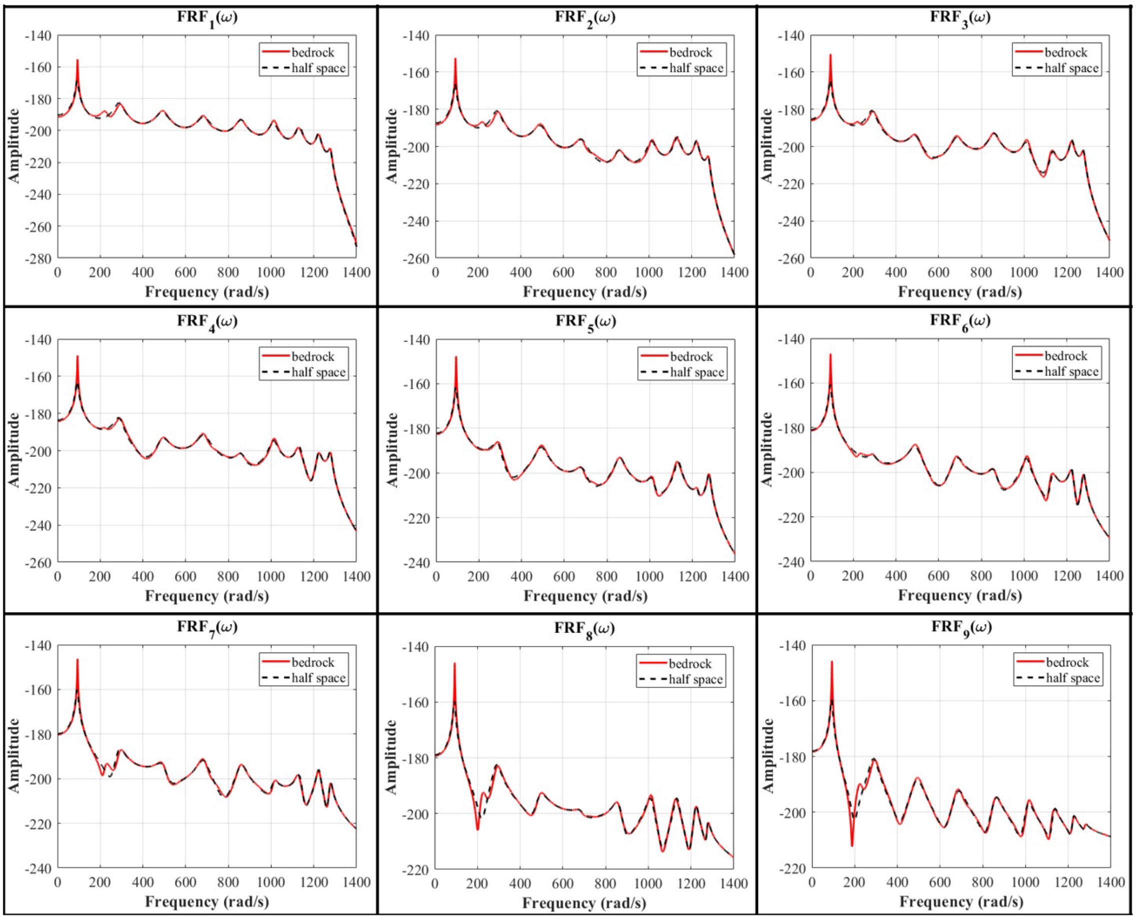

Figure 6 Structure FRFs with soil influence (System I x System II).

3.4 Extraction of modal data from both models

The Rational Fraction Polynomial Method (RFPM) was applied to this set of 9 FRFs to extract eigenfrequencies, ω_i , and modal damping coefficients, ξ_i .

Table 4 contains the values of the 9 natural frequencies and damping factors extracted for the case of the half-space. Table 5 shows the natural frequencies and damping factors for the case of the horizontal layer. It should be noted that in this second case, 12 eigenfrequencies were extracted from the FRFs by the RFP Method. As expected, the eigenfrequencies of the soil layer do influence the dynamic behavior of the soil-structure system that needs an expanded modal basis to describe its dynamics.

A comparison among the modal data from Table 3 (structure on a rigid base), Table 4 (structure on the half-space) and Table 5 (structure on a layer over bedrock) reveals that the presence of the soil introduces higher damping coefficients (Tables 4 and 5) than the structure on a fixed base (Table 3). This is consistent with the additional damping introduced in the system by the geometric damping mechanism.

Table 4 Modal parameters of System I.

Natural frequency (rad/s)	Damping Factor
ω_1=92.4	ξ_1=0.051
ω_2=289.4	ξ_2=0.064
ω_3=492.3	ξ_3=0.052
ω_4=685.3	ξ_4=0.037
ω_5=860.6	ξ_5=0.025
ω_6=1010.9	ξ_6=0.016
ω_7=1133.0	ξ_7=0.011
ω_8=1222.7	ξ_8=0.008
ω_9=1277.7	ξ_9=0.006

Table 5 Modal parameters of System II.

Natural frequency (rad/s)	Damping Factor
ω_1=94.0	ξ_1=0.010
ω_2=229.2	ξ_2=0.054
ω_3=295.5	ξ_3=0.060
ω_4=493.1	ξ_4=0.045
ω_5=589.3	ξ_5=0.026
ω_6=681.7	ξ_6=0.033
ω_7=770.4	ξ_7=0.039
ω_8=1014.0	ξ_8=0.014
ω_9=1124.5	ξ_9=0.013
ω_{10}=1137.5	ξ_{10}=0.014
ω_{11}=1222.7	ξ_{11}=0.008
ω_{12}=1277.8	ξ_{12}=0.006

3.5 Transient responses of systems with soil influence

To determine the transient responses of the structures, the previously extracted modal parameters are used as input data for the motion equations (37). The excitation force was applied at the last dof of the structure (i=9) and its time dependency is given by:

$$F(t) = \begin{cases} 0, & t < 0.01s \\ 5000N, & 0.01s \le t \le 0.05s \\ 0, & t > 0.05s \end{cases} \tag{38}$$

The motion equations were numerically integrated by the Newmark method, using the parameters given in Table 6.

Table 6 Integration parameters.

Newmark method	
Time step (s)	Δt=0.0034
Time interval (s)	0 – 1
Newmark coefficients	γ=0.25
	β=0.5

Figure 7 shows the transient responses of both systems for each dof of the structure. The validation of the present approach for the soil as a half-space model was presented in Ferraz et al. (2021b). The extension to the model of a layer over bedrock, presented in the current article is a new result. Nevertheless, the response for the structure supported by a horizontal layer over bedrock is consistent with the expected results. The amount of geometric damping of the layer is notably smaller than the one presented by the homogeneous half-space, due to the wave reflections at the rigid base. The transient response for the structure interacting with the layer shown in Figure 7 for all 9 structural dofs present a much smaller overall damping. The layer over bedrock is also slightly more rigid than the half-space resulting in a small increase in the frequency of the structure response.

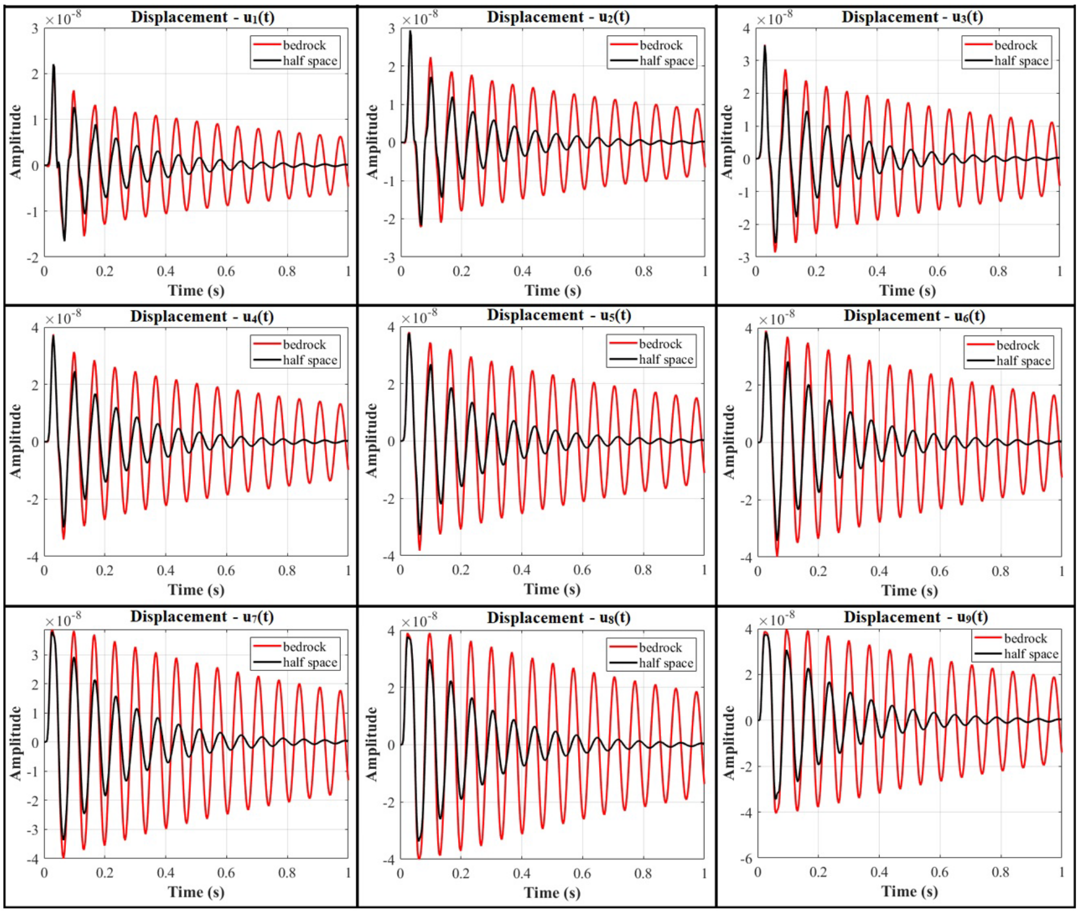

Figure 7 Transient responses (System I x System II).

4 CONCLUSION

This article presented a methodology to obtain the transient response of linear structures interaction with soil profiles. The soil response in the frequency domain was synthesized by the Boundary Element Method. A rigid and massless foundation interacting with the homogenous half-space and with a horizontal layer over a rigid stratum were the considered soil models.

The dynamic response of the structure was based on a modal description of the system, which allows to consider an arbitrary number of degrees of freedom. After the coupling of the structure with the soil, in the frequency domain, a set of Frequency Response Functions (FRFs) for the structure were synthesized in which the effects of the soil-structure interaction were already accounted for. From these modified FRFs, modal parameters were extracted in order to build a set of uncoupled differential equations that describe the coupled transient behavior of the structure. An expansion of the structural modal base was presented to properly describe the dynamics of soil-structures in which the soil model presents eigenfrequencies. The layer over a bedrock represents such a system. The results obtained for the transient response of the structures were consistent with the expected behavior of the considered cases. The methodology can be applied to arbitrary soil profiles, provided the frequency response of the soil model is available.

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