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The Equivalent Linearization Method with a Weighted Averaging for Analyzing of Nonlinear Vibrating Systems

Abstract
In this paper, the Equivalent Linearization Method (ELM) with a weighted averaging, which is proposed by Anh (Anh, 2015), is applied to analyze some vibrating systems with nonlinearities. The strongly nonlinear Duffing oscillator with third, fifth, and seventh powers of the amplitude, the other strongly nonlinear oscillators and the cubic Duffing with discontinuity are considered. The results obtained via this method are compared with the ones achieved by the Min-Max Approach (MMA), the Modified Lindstedt – Poincare Method (MLPM), the Parameter – Expansion Method (PEM), the Homotopy Perturbation Method (HPM) and 4^{th} order Runge-Kutta method. The obtained results demonstrate that this method is very convenient for solving nonlinear equations and also can be successfully exerted to a lot of practical engineering and physical problems.

Keywords
nonlinear oscillator, Equivalent Linearization Method, weighted averaging.

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1 INTRODUCTION

Nonlinear oscillations systems are such phenomena that mostly occur nonlinearly. These systems are important in engineering because many practical engineering components consist of vibrating systems that can be modeled using oscillator systems such as elastic beams supported by two springs or mass-on-moving belt or nonlinear pendulum and vibration of a milling machine. Hence solving of governing equations and due to a limitation of existing exact solutions have been one of the most time-consuming and difficult affairs among researchers of vibrations.

The amplitude–frequency relationship is of significant importance for the accurate prediction of nonlinear oscillator systems in many areas of physics and engineering, especially in nonlinear structural dynamics. Therefore, the analyzing of nonlinear systems has been widely considered. In recent years, many powerful methods are used to find approximate solution as well as the amplitude-frequency relationship to the nonlinear differential equations. Some of these methods are Homotopy Perturbation Method (HPM) (He, 1999; He, 2004a; He, 2004b; He, 2004c; Turgut et al., 2007; Bayat et al., 2012), Max-Min Approach (MMA) (He, 2008; Ganji et al., 2010; Chen et al., 2011; Dumaz et al., 2011; Yazdi et al., 2012; Bayat et al., 2012), Variational Iteration Method (VIM) (Bayat et al., 2012), Energy Balance Method (EBM) (Ganji et al., 2009; Khah et al., 2010; Younesian et al., 2010; Bayat et al., 2012), Amplitude-Frequency Formulation (AFF) (Chen et al., 2011; Jouyburi et al., 2014; Bayat et al., 2012), Parameter Expansion Method (PEM) (Kayaa et al., 2009; Dumaz et al., 2011; Darvishia et al., 2008; Zhao, 2009; Bayat et al., 2012), Homotopy Analysis Method (HAM) (He, 2004c; Bayat et al., 2012, Shahram Shahlaei-Far et al., 2016), Modified Homotopy Perturbation Method (MHPM) (Jouybari et al., 2014), Equivalent linearization Method (ELM) (Krylov et al., 1943; Caughey, 1959; Iyengar, 1988; Anh et al., 1995; Anh et al., 1997; Elishakoff et al., 2009; Anh, 2015) and combining Newton's Method with the Harmonic Balance Method (Lim et al., 2006).

The Equivalent Linearization Method of Kryloff and Bogoliubov (Krylov et al., 1943) was generalized to the case of nonlinear dynamic systems with random excitation by Caughey (Caughey, 1959). And then, this method has been developed by many authors (Iyengar, 1988; Anh et al., 1995; Anh et al., 1997; Elishakoff et al., 2009). It has been shown that the Gaussian equivalent linearization is presently the simplest tool widely used for analyzing nonlinear stochastic problems. Nevertheless, the accuracy of the Equivalent Linearization Method with conventional averaging normally reduces for middle or strong nonlinear systems. A reason is that some terms will vanish in the averaging process, for example the averaging value of the functions *sin(t)* and *cos(t)* over one period is equal to zero. Anh N. D. (Anh, 2015) proposed a new way for determining averaging values, instead of using conventional averaging process author introduced weighted coefficient functions.

In this paper, the equivalent linearization method with weighted averaging is applied to nonlinear oscillators. To illustrate the applicability and accuracy of the method, four examples are presented: nonlinear Duffing oscillator with third, fifth, and seventh powers of the amplitude, the strongly nonlinear oscillators and the cubic Duffing with discontinuity. The amplitude–frequency relationship can be readily obtained by this method. The results compared with the ones given by the numerical method and other well-known techniques show the accuracy of this method.

2 THE EQUIVALENT LINEARIZATION METHOD WITH A WEIGHTED AVERAGING

2.1 The Equivalent Linearization Method

In order to present the general idea of the equivalent linearization method, we consider a nonlinear oscillator governed by the following equation:

$$\ddot{X} + 2h\dot{X} + \omega_0^2 X + g(\dot{X}, X) = 0 \tag{1}$$

where $g(\dot{X},X)$ is a nonlinear function only depending on two variables of velocity $\dot{X}(t)$ and displacement $X(t)$, h and ω_0 are constants. The corresponding equivalent linear oscillator is described by the equation as follows:

$$\ddot{X} + (2h + \mu)\dot{X} + (\omega_0^2 + \lambda)X = 0 \tag{2}$$

The equation error between the two oscillators is taken as

$$e(\dot{X},X)=g(\dot{X},X)-\mu\dot{X}-\lambda X \tag{3}$$

The coefficients of linearization in the linearized Eq. (3) are found from a certain optimal criterion. There are some criteria for determining these coefficients. The most common criterion is the mean square error criterion which requires the mean square of equation error to be minimum:

$$\left\langle e^2(\dot{X},X)\right\rangle=\left\langle\left(g(\dot{X},X)-\mu\dot{X}-\lambda X\right)^2\right\rangle\to\underset{\mu,\lambda}{Min} \tag{4}$$

Thus, from

$$\frac{\partial}{\partial\lambda}\left\langle e^2(\dot{X},X)\right\rangle=0$$
$$\frac{\partial}{\partial\mu}\left\langle e^2(\dot{X},X)\right\rangle=0$$

it follows that

$$\lambda=\frac{\langle gX\rangle\langle\dot{X}^2\rangle-\langle g\dot{X}\rangle\langle X\dot{X}\rangle}{\langle X^2\rangle\langle\dot{X}^2\rangle-\langle X\dot{X}\rangle^2} \tag{5a}$$

$$\mu=\frac{\langle g\dot{X}\rangle\langle X^2\rangle-\langle gX\rangle\langle X\dot{X}\rangle}{\langle X^2\rangle\langle\dot{X}^2\rangle-\langle X\dot{X}\rangle^2} \tag{5b}$$

In the formulas in Eqs. (4) and (5), the symbol $\langle\bullet\rangle$ denotes the time-averaging operator in classical meaning:

$$\left\langle f(t)\right\rangle=\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(t)dt \tag{6}$$

For a ω-frequency function $f(\omega t)$, the averaging process is taken during one period T, i.e.

$$\left\langle f(\omega t)\right\rangle=\frac{1}{T}\int_0^T f(\omega t)dt=\frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau,\ \ \tau=\omega t \tag{7}$$

In this technique, the importance of the attended terms is considered as the same on time scale. In fact, their roles generally differ from time to time. That may be one of the reasons causing the classical equivalent replacement be effective only for oscillators with weak nonlinearity, but normally not good for ones with strong nonlinearity. In order to improve this shortcoming, the averaging operation with weighting functions is proposed in the next section. This idea is introduced by Anh, N. D (Anh, 2015).

2.2 The Weighted Averaging

It is well-known that for a given data set the most common statistic is the arithmetic mean. The concept for the average of a data set can be extended to functions. The conventional average value of an integrable deterministic function $x(t)$ on a domain D: $(0,d)$ is a constant value defined by:

$$<x(t)>=\frac{1}{d}\int_0^d x(t)dt \tag{8}$$

In many cases when the function $x(\omega t)$ is periodic with period $2\pi/\omega$, the value d is taken as $2\pi/\omega$ and it leads to the averaged value of $x(t)$ over one period:

$$<x(\omega t)>=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(\omega t)dt=\frac{1}{2\pi}\int_0^{2\pi} x(\tau)d\tau \tag{9}$$

where $\tau=\omega t$ is the new variable or "new time". Averaged values play surely major roles in the past and at present, however, the definition (8) has some deficiencies, for example, if (8) or (9) are equal zero, the information about $x(t)$ will be lost. For all harmonic functions $cos(n\omega t)$ and $sin(n\omega t)$, this observation is true. The dual approach to averaged values may be a possible way to suggest an alternative choice for the conventional average value, namely the constant coefficient $1/d$ in Eq. (8) can be extended to a weighted coefficient as a function $h(t)$. Thus one gets so-called a weighted average value:

$$W(x(t))=\int_0^d h(t)x(t)dt \tag{10}$$

where the condition of normalization is satisfied:

$$\int_0^d h(t)dt=1 \tag{11}$$

There are three basic weighted coefficients:

+ *Basic optimistic weighted coefficients*: They are increasing functions of t and denoted as $O(t)$. Examples are αt^{β} and $\alpha e^{\beta t}$, $a, \beta > 0$.

+ *Basic pessimistic weighted coefficients*: They are decreasing functions of t and denoted as $P(t)$. Examples are αt^{β} and $\alpha e^{\beta t}$, $a < 0, \beta > 0$; or $a > 0, \beta < 0$.

+ *Neutral weighted coefficients*: They are denoted as $N(t)$ and are constants.

An arbitrary weighted coefficient $h(t)$ can be obtained as summation and/or product of basic weighted coefficients. Example is:

$$h(t)=\sum_{i=1}^{n} A_iO_i(t)+B_iP_i(t)+C_iO_i(t)P_i(t)+N(t) \tag{12}$$

where A_i, B_i, C_i are constant.

In this paper, we will consider only ω-periodic functions $x(\omega t)$. A special form of weighting coefficient is introduced as:

$$h(t)=s^2\omega^2te^{-s\omega t}, s>0 \tag{13}$$

where s is constant.

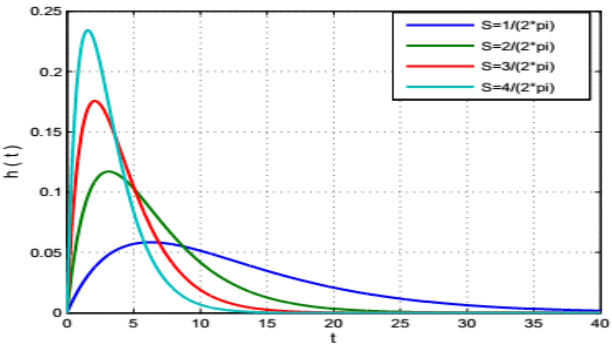

Figure 1: Plot of $h(t) = s^2 t e^{-st}$.

It is seen that the weighting coefficient (13), obtained as a product of the optimistic weighting coefficient t and the pessimistic weighting coefficient $e^{-s\omega t}$, has one maximal value at $t_{\max} = 1/(\omega s)$, and then decreases to zero as $t \to \infty$ (see Fig. 1). If one requires that the time t_{max} is equal to $T/n=2\pi/(n\omega)$ where n is a natural number or zero, we get $s=n/(2\pi)$. So the meaning of s can be specified as follows: for $n = 1$, $s=1/(2\pi)$ the weighting coefficient (13) has maximal value after one period, and for $n=4$, $s=4/(2\pi)$ the weighting coefficient (13) has maximal value after quarter period, and for $n=0$, $s=0$ the weighting coefficient (13) has maximal value at infinity. This case corresponds to the conventional averaged value.

Based on the weighting coefficient (13), a new weighted average value is proposed:

$$\langle x(\omega t)\rangle = \int_0^\infty s^2\omega^2 t e^{-s\omega t} x(\omega t)dt = \int_0^\infty s^2 \tau e^{-s\tau} x(\tau)d\tau \tag{14}$$

which is a linear operator. From Laplace transformation, we get, for example:

$$\langle \cos(n\omega t)\rangle = \int_0^\infty s^2\omega^2 t e^{-s\omega t} \cos(n\omega t)dt = \int_0^\infty s^2 \tau e^{-s\tau} \cos(n\tau)d\tau = s^2 \frac{s^2 - n^2}{(s^2+n^2)^2} \tag{15}$$

$$\langle \sin(n\omega t)\rangle = \int_0^\infty s^2\omega^2 t e^{-s\omega t} \sin(n\omega t)dt = \int_0^\infty s^2 \tau e^{-s\tau} \sin(n\tau)d\tau = s^2 \frac{2sn}{(s^2+n^2)^2} \tag{16}$$

As ω-periodic functions $x(\omega t)$ can be expanded into Fourier series, hence we can easy calculate (14) by using Eqs. (15) and (16).

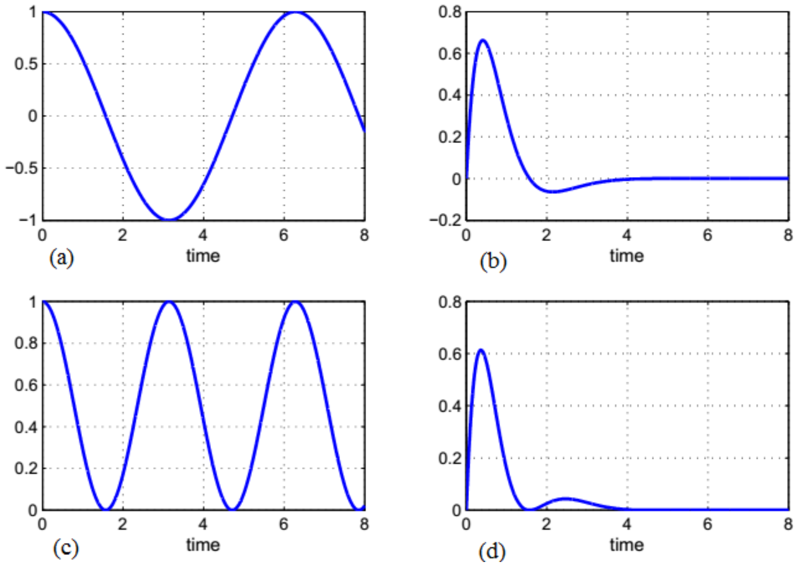

Figure 2: Graphs of the function: (a) - $cos(\tau)$, (b) - $h(\tau)cos(\tau)$, (c) – $cos^2(\tau)$, and (d) - $h(\tau)cos^2(\tau)$.

The proposed averaging operation can preserve the linear properties of the classical one. Furthermore, it can conserve some terms which vanish in the classical averaging process. The effect of the weighted function to the averaging process can be recognized, for instance, when we observe the graphs of functions $cos(\tau)$, $h(\tau)cos(\tau)$, $cos^2(\tau)$, and $h(\tau)cos^2(\tau)$ in Fig. 2. The function $h(\tau)$ adjusts the value of the functions $cos(\tau)$ and $cos^2(\tau)$, maintains partly the periodicities of the functions $cos(\tau)$ and $cos^2(\tau)$, also condenses these function values in the first period, gives a weight in the first half of the first period, reduces the difference maximum and minimum values as well as regulates the functions during the period. These adjustments may make a positive effect on the averaging process. Therefore, the linearized equation replacement for the original one may be better in some senses.

In this paper, for the sake of computation convenience, the parameter s is chosen equal to 2.

3 SOME EXAMPLES AND DISCUSSIONS

3.1 Example 1

We consider the strongly nonlinear Duffing oscillator with third-, fifth-, and seventh-order nonlinear terms in the following form:

$$\ddot{u} + u + \alpha u^3 + \beta u^5 + \gamma u^7 = 0 \tag{17}$$

with the initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{18}$$

The linearized equation of Eq. (17) is:

$$\ddot{u} + (1 + k)u = 0 \tag{19}$$

The equation error between the two Eqs. (17) and (19) is:

$$e(u) = \alpha u^3 + \beta u^5 + \gamma u^7 - ku \tag{20}$$

The unknown coefficient k is determined from the mean square error criterion

$$\frac{\partial}{\partial k}\left\langle e^2(u)\right\rangle = 0$$

it yields:

$$k = \frac{\alpha\left\langle u^4\right\rangle + \beta\left\langle u^6\right\rangle + \gamma\left\langle u^8\right\rangle}{\left\langle u^2\right\rangle} \tag{21}$$

The periodic solution and the frequency of Eq. (19) are:

$$u(t) = A\cos(\omega t), \qquad \omega = \sqrt{1+k} \tag{22}$$

Now, we calculate the averaging operators in Eq. (21) by using Eq. (14):

$$\left\langle u^2\right\rangle = \left\langle A^2\cos^2(\omega t)\right\rangle = A^2\,\frac{s^4+2s^2+8}{(s^2+4)^2} \tag{23}$$

$$\left\langle u^4\right\rangle = \left\langle A^4\cos^4(\omega t)\right\rangle = A^2\,\frac{248s^4+416s^2+1536+28s^6+s^8}{(s^2+4)^2(s^2+16)^2} \tag{24}$$

$$\left\langle u^6\right\rangle = \left\langle A^6\cos^6(\omega t)\right\rangle = A^6\,\frac{1658880+440064s^2+282496s^4+45712s^6+3168s^8+94s^{10}+s^{12}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2} \tag{25}$$

$$\begin{aligned}\left\langle u^8\right\rangle &= \left\langle A^8\cos^8(\omega t)\right\rangle = \\ &= A^8\,\frac{1516142592s^2+1014806528s^4+192596992s^6+17013120s^8+5945425920+768000s^{10}+18256s^{12}+216s^{14}+s^{16}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2(s^2+64)^2}\end{aligned} \tag{26}$$

In case s = 2, substituting Eqs. (23), (24), (25) and (26) into Eq. (21), and then substituting Eq. (21) into Eq. (22) we get the approximate frequency and solution of this oscillator as follows:

$$\omega = \sqrt{1+0.72\alpha A^2+0.575\beta A^4+0.4836\gamma A^6} \tag{27}$$

and

$$u(t) = A\cos\left(\sqrt{1+0.72\alpha A^2+0.575\beta A^4+0.4836\gamma A^6}\; t\right) \tag{28}$$

The frequencies $\omega_{present}$ calculated from the proposed method, the frequencies ω_{MMA} obtained by the Min-Max Approach (Yazdi et al., 2012) are compared with the exact ones ω_e in Table 1 and in Figs. 3–4 for different values of the oscillation amplitude. It can be seen from Table 1 that the approximate frequencies $\omega_{present}$ are closer to the exact frequencies ω_e than the one ω_{MMA}.

The numerical results obtained by three different methods are illustrated in Figs. 3-4. As shown in Figs. 3 and 4, the validity of the solution technique is guaranteed even for stronger nonlinearities.

The approximate frequency is obtained by using the Min-Max Approach given by Yazdi et al. (Yazdi et al., 2012) as follows:

$$\omega_{MMA} = \sqrt{1 + \frac{3}{4}\alpha A^2 + \frac{5}{8}\beta A^4 + \frac{35}{64}\gamma A^6} \tag{29}$$

The exact frequency of this oscillator as follows (Younesian et al., 2010):

$$\omega_e = 2\pi \left[4\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\frac{1}{2}\left(1+\sin^2\theta\right)\alpha A^2 + \frac{1}{3}\left(1+\sin^2\theta+\sin^4\theta\right)\beta A^4 + \frac{1}{4}\left(1+\sin^2\theta+\sin^4\theta+\sin^6\theta\right)\gamma A^6}} \right]^{-1} \tag{30}$$

α	β	γ	A	ω_e	ω_{MMA}	Error (%)	$\omega_{Present}$	Error (%)
1	1	1	0.1	1.0037732	1.0037744	0.000119	1.0036224	0.015020
5	5	5	0.1	1.0187037	1.0187321	0.002795	1.0179833	0.070721
5	5	5	0.5	1.4633113	1.4749702	0.796748	1.4551515	0.557625
10	10	10	0.5	1.8060216	1.8305939	1.360579	1.7985916	0.411399
10	10	10	1	4.3059814	4.4965264	4.425124	4.3342404	0.656274
50	50	50	1	9.3991494	9.8536161	4.835189	9.4830481	0.892619

Table 1: A comparison between the natural frequencies with various parameters for Example 1.

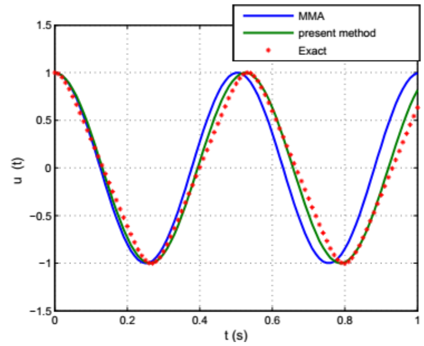

Figure 3: A comparison between the approximate and exact solutions for Example 1, with α = 50, β = 100, γ = 100, A = 1.

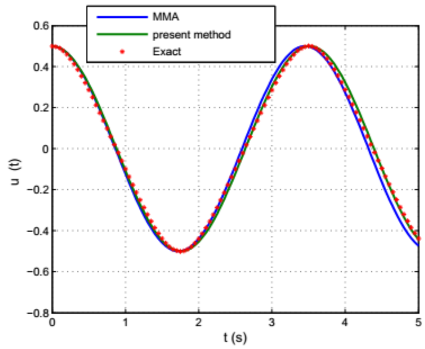

Figure 4: A comparison between the approximate and exact solutions for Example 1, with α = 10, β = 10, γ = 5, A = 0.5.

3.2 Example 2

We consider the following nonlinear oscillator (He, 2002):

$$(1+\varepsilon u^2)\ddot{u}+u=0, \quad u(0)=A, \ \dot{u}(0)=0 \tag{31}$$

The linearized equation of Eq. (31) is:

$$\ddot{u}+\omega^2 u=0 \tag{32}$$

The equation error between the two Eqs. (31) and (32) is:

$$e(u)=(1+\varepsilon u^2)\ddot{u}+u-\ddot{u}-\omega^2 u=\varepsilon u^2\ddot{u}+u-\omega^2 u \tag{33}$$

where ω^2 is determined by using the mean-square criterion, as follows:

$$\omega^2=\frac{\langle u^2\rangle+\varepsilon\langle u^3\ddot{u}\rangle}{\langle u^2\rangle} \tag{34}$$

The periodic solution of linearized Eq. (32) is:

$$u(t)=A\cos(\omega t) \tag{35}$$

Using the definition (14), we calculate averaging operators in Eq. (34):

$$\begin{aligned}\langle u^2\rangle=\langle A^2\cos^2\omega t\rangle&=\int_0^{+\infty}A^2 s^2\omega^2 t e^{-s\omega t}\cos^2(\omega t)\,dt\\&=\int_0^{+\infty}A^2 s^2\tau e^{-s\tau}\cos^2(\tau)\,d\tau=A^2\frac{s^4+2s^2+8}{(s^2+4)^2}\end{aligned} \tag{36}$$

$$\begin{aligned}\langle u^3\ddot{u}\rangle=\langle -A^4\omega^2\cos^4\omega t\rangle&=-A^4\omega^2\int_0^{+\infty}s^2\omega^2 t e^{-s\omega t}\cos^4(\omega t)\,dt\\&=-\int_0^{+\infty}A^4\omega^2 s^2\tau e^{-s\tau}\cos^4(\tau)\,d\tau=-A^4\omega^2\frac{248s^4+416s^2+1536+28s^6+s^8}{(s^2+4)^2(s^2+16)^2}\end{aligned} \tag{37}$$

With s is chosen equal to 2, substituting Eqs. (36) and (37) into Eq. (34), we get:

$$\omega^2=1-\varepsilon A^2\frac{9216}{12800}\omega^2 \tag{38}$$

From Eq. (38), we get the approximate frequency of this oscillator:

$$\omega = \frac{1}{\sqrt{1+0.72\varepsilon A^2}} \tag{39}$$

And thus, the approximate solution of this oscillator is:

$$u(t) = A\cos\left(\frac{1}{\sqrt{1+0.72\varepsilon A^2}}t\right) \tag{40}$$

To illustrate the remarkable accuracy of the obtained results, we compare the approximate period

$$T = 2\pi\sqrt{1+0.72\varepsilon A^2} \tag{41}$$

with the approximate period abtained by Modified Lindstedt-Poincare method (MLPM) (He, 2002)

$$T_{MLPM} = 2\pi\sqrt{1+\frac{3}{4}\varepsilon A^2} \tag{42}$$

and the exact one (He, 1999)

$$T_{ex} = 4\sqrt{\pi}\int_0^A \frac{du}{\sqrt{\ln(1+\varepsilon A^2)-\ln(1+\varepsilon u^2)}} \tag{43}$$

In case $\varepsilon A^2 \to \infty$, Eq. (43) reduces to (He, 2002):

$$\lim_{\varepsilon A^2\to\infty} T_{ex} = 2\sqrt{2\pi\varepsilon}A \tag{44}$$

So for large ε, it follows:

$$T_{ex} \sim \sqrt{\varepsilon}A \tag{45}$$

It is obvious that the approximate periods (41) and (42) have the same feature as the exact one for $\varepsilon \gg 1$. And in case $\varepsilon \to \infty$, we have

$$\lim_{A\to\infty}\frac{T_{ex}}{T} = \frac{2\sqrt{2\pi\varepsilon}A}{2\pi\sqrt{0.72\varepsilon}A} = 0.9403 \tag{46}$$

and

$$\lim_{A\to\infty}\frac{T_{ex}}{T_{MLPM}}=\frac{2\sqrt{2\pi\varepsilon A}}{\pi\sqrt{3\varepsilon A}}=0.9213 \tag{47}$$

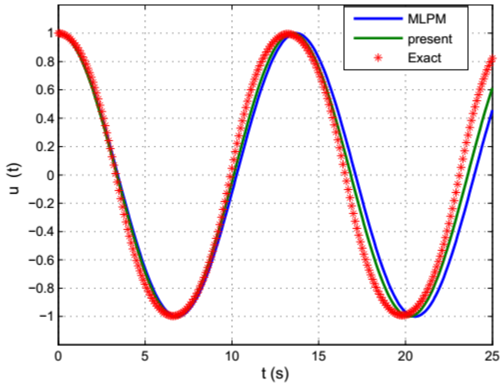

Figure 5: Comparison of time history diagram of displacement between the Present, MLPM and Exact solutions at $\varepsilon=5,\ A=1$.

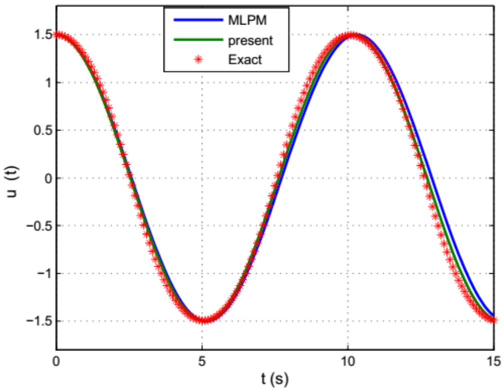

Figure 6: Comparison of time history diagram of displacement between the Present, MLPM and Exact solutions at $\varepsilon=1,\ A=1.5$.

Therefore, for any values of ε, it can be easily proved that the maximal relative error is less than 6.349% for this method and 8.54% for Modified Lindstedt-Poincare method on the whole solution domain $(0<\varepsilon<\infty)$.

The numerical results obtained by three different methods are illustrated in Figs. 5-6 for different values of ε and A. Numerical results validate the gain accuracy of this method.

3.3 Example 3

We consider the following nonlinear oscillator (Darvishia et al., 2008; Lim et al., 2006):

$$\ddot{u}+\frac{u^3}{1+u^2}=0,\quad u(0)=A,\ \ \dot{u}(0)=0 \tag{48}$$

The Eq. (48) can be written as follows:

$$(1+u^2)\ddot{u}+u^3=0 \tag{49}$$

The linearized equation of Eq. (49) is:

$$\ddot{u}+\omega^2 u=0 \tag{50}$$

The equation error between the two Eqs. (49) and (50) is:

$$e(u)=(1+u^2)\ddot{u}+u^3-\ddot{u}-\omega^2 u=u^2\ddot{u}+u^3-\omega^2 u \tag{51}$$

where ω^2 is determined by using the mean-square criterion, as follows:

$$\omega^2=\frac{\langle \ddot{u}u^3\rangle+\langle u^4\rangle}{\langle u^2\rangle} \tag{52}$$

The periodic solution of linearized Eq. (50) is:

$$u(t)=A\cos(\omega t) \tag{53}$$

Using the solution (53), we calculate averaging operators in Eq. (52), then substituting these operators into Eq. (52) and with note that parameter s is chosen equal to 2, we get the approximate frequency of this oscillator:

$$\omega=\frac{\sqrt{0.72A^2}}{\sqrt{1+0.72A^2}} \tag{54}$$

Thus, the approximate solution of this oscillator is:

$$u(t)=A\cos\left(\frac{\sqrt{0.72A^2}}{1+0.72A^2}t\right) \tag{55}$$

A	ω_{ex}	ω_{PEM}	R. Error (%)	$\omega_{Present}$	R. Error (%)
0.01	0.00847	0.00866	2.24321	0.00848	0.11806
0.05	0.04232	0.04326	2.22117	0.04239	0.16541
0.1	0.08439	0.08628	2.23960	0.08455	0.18959
0.5	0.38737	0.39736	2.57893	0.39057	0.82608
1	0.63678	0.65465	2.80631	0.64699	1.60338
5	0.96698	0.97435	0.76217	0.97333	0.65668
10	0.99092	0.99339	0.24926	0.99313	0.22303

Table 2: Comparison of the approximate frequencies with the exact frequencies.

Comparison of the approximate frequencies ω in Eq. (54) and the approximate frequencies obtained by Parameter-Expansion Method (PEM) ω_{PEM} (Bayat et al., 2012) in Eq. (56) with exact frequencies ω_{ex} in Eq. (57) is tabulated in Table 2. Table 2 shows that the maximum relative error is less than 1.60338% for this method and 2.80631% for Parameter-Expansion Method.

The approximate frequency obtained by PEM as follows (Bayat et al., 2012):

$$\omega_{PEM} = \frac{\sqrt{3A^2}}{\sqrt{4+3A^2}} \tag{56}$$

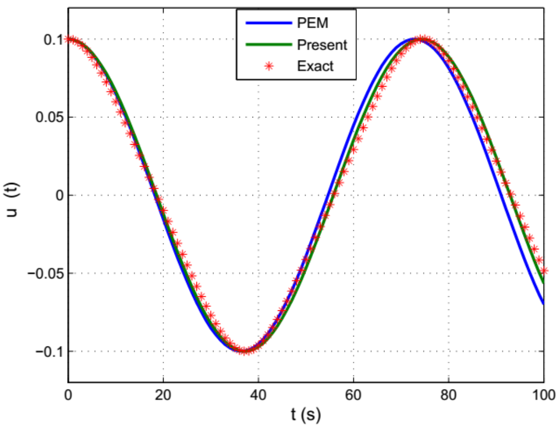

Figure 7: Comparison of time history diagram of displacement between the Present, PEM and Exact solutions at *A=0.1*.

The exact frequency of this oscillator is (Ganji et al., 2010):

$$\omega_{ex} = \frac{2\pi}{4\int_0^{\pi/2} \left\{ \dfrac{A^2\cos^2(\theta)}{A^2\cos^2(\theta) + \ln\left[1 - \dfrac{A^2\cos^2(\theta)}{1+A^2}\right]} \right\}^{1/2} d\theta} \tag{57}$$

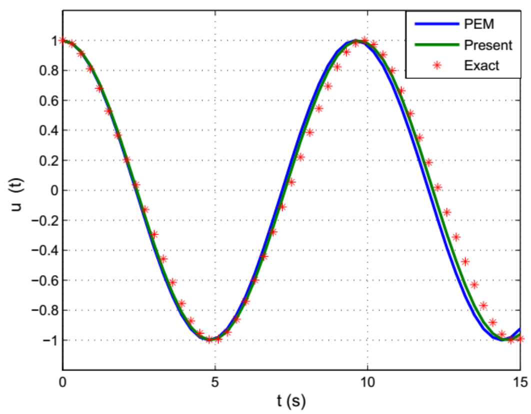

Figure 8: Comparison of time history diagram of displacement between the Present, PEM and Exact solutions at *A=1*.

The accuracy of the solution obtained this method can be observed in Figs. 7-8 which represent comparisons of analytic solutions of *u(t)* based on time for this method and the one obtained by Parameter-Expansion Method as well as with the exact solution.

3.4 Example 4

We consider the Duffing oscillator with discontinuity (He, 2004a):

$$\ddot{u} + \beta u^3 + \varepsilon u|u| = 0 \tag{58}$$

with the initial conditions:

$$u(0) = A, \qquad \dot{u}(0) = 0 \tag{59}$$

The linearized equation of Eq. (58) is:

$$\ddot{u} + \alpha u = 0 \tag{60}$$

The equation error between the two Eqs. (58) and (60) is:

$$e(u) = \beta u^3 + \varepsilon u|u| - \alpha u \tag{61}$$

The unknown coefficient α is determined from the mean square error criterion

$$\frac{\partial}{\partial \alpha}\left\langle e^2(u)\right\rangle = 0$$

it follows that:

$$\alpha = \frac{\beta\left\langle u^4\right\rangle + \varepsilon\left\langle u^2|u|\right\rangle}{\left\langle u^2\right\rangle} \tag{62}$$

The priodic solution and the frequency of Eq. (60) are:

$$u(t) = A\cos(\omega t), \qquad \omega = \sqrt{\alpha} \tag{63}$$

It is similar to Example 1, Example 2 and Example 3, we calculate averaging operators $\left\langle u^2\right\rangle$, $\left\langle u^2|u|\right\rangle$ and $\left\langle u^4\right\rangle$; and then substituting these operators into Eq. (62), yields the approximate frequency:

$$\omega = \sqrt{0.8324\varepsilon A + 0.72\beta A^2} \tag{64}$$

and the approximate solution:

$$u(t) = A\cos\left(\sqrt{0.8324\varepsilon A + 0.72\beta A^2}\; t\right) \tag{65}$$

Accuracy of the approach for this example is shown in Figs. 9–11. We performed a comparison between the results obtained by this method, the ones obtained by He (He, 2004a) using the Homotopy Perturbation Method and outcomes achieved using Runge-Kutta 4^{th} order for different values of A, β and ε.

Figs. 9-11, with the small, middle and large values of β and ε, show that the results obtained by the present method are more exact than the ones obtained by the homotopy perturbation method.

We compare the approximate period obtained by this method T with the one obtained by the Homotopy Perturbation Method T_{HPM} .

The approximate period of this oscillator is:

$$T = \frac{2\pi}{\sqrt{0.8324\varepsilon A + 0.72\beta A^2}} \tag{66}$$

The approximate period obtained by the Homotopy Perturbation Method (He, 2004a) is:

$$T_{HPM} = \frac{2\pi}{\sqrt{\frac{8}{3\pi}\varepsilon A + \frac{3}{4}\beta A^2}} \tag{67}$$

In case $\varepsilon=0$, these periods can be written as:

$$T = \frac{2\pi}{\sqrt{0.72\beta A^2}} = 7.405\beta^{-1/2}A^{-1} \tag{68}$$

and

$$T_{HPM} = \frac{4\pi}{\sqrt{3\beta A^2}} = 7.255\beta^{-1/2}A^{-1} \tag{69}$$

The exact period can be readily obtained, which reads (Acton et al., 1985):

$$T_{ex} = 7.416\beta^{-1/2}A^{-1} \tag{70}$$

Thus, the maximal relative error of T_{HMP} is less than 2.2% and the maximal relative error of T is less than 0.15% for all $\beta>0$.

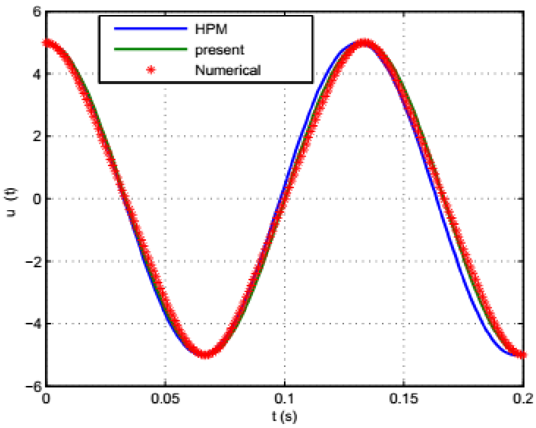

Figure 9: A comparison between the approximate and Runge–Kutta solutions for Example 3, β = 100, ε = 100, A = 5.

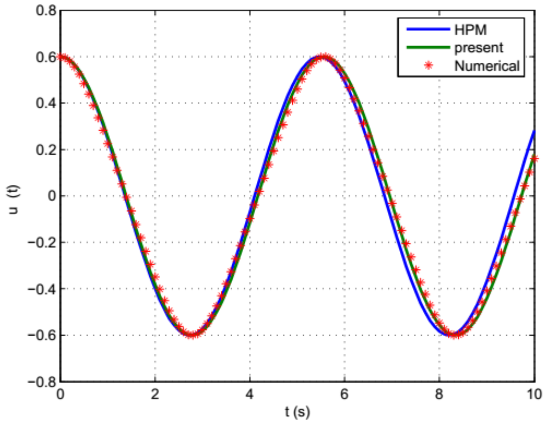

Figure 10: A comparison between the approximate and Runge–Kutta solutions for Example 3, $\beta = 3$, $\varepsilon = 1$, $A = 0.6$.

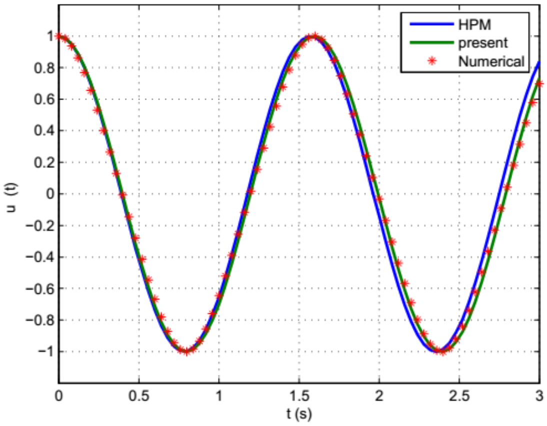

Figure 11: A comparison between the approximate and Runge–Kutta solutions for Example 3, $\beta = 3$, $\varepsilon = 1$, $A = 1$.

4 CONCLUSIONS

In this paper, the equivalent linearization method with weighted averaging is applied to analyze the nonlinear oscillation systems. This method is proposed by Anh in 2015. The accuracy of this method is investigated by four nonlinear oscillation systems. The results show that this method is useful to obtain analytical solutions for oscillators and vibration problems with nonlinearities. And the results indicate that the solution procedure is easy and provide a remarkable accuracy. However, the value of the parameter s in the express of weighted coefficient $h(t)$ should be chosen to give better and the best solution is still required further investigation.

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