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Behavioral Study of Finite Beam Resting on Elastic Foundation and Subjected to Travelling Distributed Masses

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Abstract

The classical problem of the response characteristics of uniform structural member resting on elastic subgrade and subjected to uniform partially distributed load is studied in this work. The closed form solutions of the governing fourth order partial differential equations with variable coefficients are presented using an elegant analytical technique for the moving force and mass models. Various results and analyses are carried out on each of the pertinent boundary conditions and phenomenon of resonance is studied for the dynamical system. It was found that in all illustrative examples considered, for the same natural frequency, the critical speed for moving distributed mass problem is smaller than that of the moving distributed force problem. Hence, resonance is reached earlier in moving mass beam-load interaction problem. Finally, this work has suggested valuable methods of analytical solution for this category of problems for all boundary conditions of practical interest.

Keywords

Dynamic characteristics, Resonance, Subgrade, Distributed Loads, Dynamical system.

1 INTRODUCTION

Analyses of the dynamic characteristics of elastic structural members resting on elastic subgrade, such as railway, tracks, highway pavements, navigation locks and structural foundations constitute an important part of the civil Engineering, Mathematical Physics and other related fields. These elastic structures are very useful in many fields of research, thus their dynamic behaviours when under the action of travelling loads of different forms have received extensive attention in the open literature [Oni and Omolofe (2005a), Hassan et al (2016), Omolofe (2013), Ismail (2015)]. When these important engineering structures are resting on an elastic foundation, the structure-foundation interaction effects play significant roles in their response behaviour and alter the dynamic states of the structures from those vibrating in the absence of foundation [Ugurlu et al (2008)]. Hence, the dynamic behaviour of structures on elastic foundation is of great importance in structural, aerospace, civil, mechanical and marine engineering applications.

Consequently, it is important to clarify the influence of the foundation on the behaviour of elastic structures in engineering designs. Furthermore, to accurately assess the dynamic response of any structural member on elastic foundations, a mechanical model is required to predict the interaction effects between such structures and foundations. Beams on elastic foundation and under the actions of the moving loads have received a considerable attention in literature; see for example references [Clastornik et al (1986), Thambiratnam and Zhuge (1996), Sun and Luo (2008), Ying et al (2008)] However, most of these works employed the simplest mechanical model which was developed by Winkler and generally referred to as a one-parameter model. The deficiency of this model is that it assumes no interaction between the springs, so it does not accurately represent the characteristics of many practical foundations [Eisenberger and Clastornik (1987)]. Thus to overcome the deficiencies inherent in Winkler formulation, a two-parameter foundation models which takes into account the effect of shear interactions between springs has been suggested.

In general, such analyses are mathematically complex due to the difficulty often encounter in modeling the mechanical response of the subgrade which is governed by many factors. When these structures are acted upon by moving loads, the dynamic analyses of the system become much more complicated.

It is known from earlier studies that, the problem of assessing transverse vibrations of elastic solid structures subjected to moving loads has been commonly considered for a point-like type of moving load, see for examples [Gbadeyan and Oni (1992), Sadiku and Leipholz (1989), Lee (1994), Oni and Omolofe (2005b), Oni and Awodola (2005)]. While studies concerning a dynamical system involving moving distributed loads are not so common. However, in engineering practice moving loads are most often in the form of distributed mass rather than that of moving lumped mass. When the moving load is distributed, the problem of investigating the load-structure interactions becomes much more complicated. Thus, to study the dynamic characteristics of such dynamical systems to the degree of aceptable accuracy required and also for practical purposes, it is useful to consider elastic structural members subjected to moving distributed loads.

Among few authors in recent times who made effort to tackle the problem of elastic structures carrying distributed moving masses are [Esmailzadeh and Ghorashi (1995)] who carried out an analysis of the dynamic behaviour of Bernoulli-Euler beam carrying uniform partially distributed moving masses. They solved the problem by means of conventional analytical technique, which is only suitable for the simple horizontal beam and will suffer much difficult if the structures are complicated. In this study, the convective terms which describe the dynamic effects of the moving mass were omitted. This approximation is not generally reasonable unless the mass moves at very low speed, and it may lead to significant errors in the evaluation of the system response. Others include, [Dada (2002)] who worked on the vibration analysis of elastic plates under uniform partially distributed loads and [Adetunde (2003)] who studied the dynamical response of Rayleigh beam carrying added mass and traversed by uniform partially distributed moving loads. However, in these studies numerical simulations were employed.

More recently, [Andi (2013), Ogunyebi (2006), Andi and Oni (2014)] carried out dynamical analysis of structural members carrying uniform partially distributed masses with general boundary conditions under travelling distributed loads. In these studies, versatile analytical techniques were used to obtain solutions valid for all variants of classical boundary conditions. Though these authors presented a very good analysis of the response of beams to distributed loads, but their studies failed to represent the physical reality of the problem formulation as they employed in their studies a simplified model of the distributed load in which the factor that measures the degree of the load distribution was omitted.

Thus, this work therefore concerns the problem of the behavioral study of a slender member continuously supported by elastic subgrade and subjected to uniform partially distributed moving masses and sets at solving this class of dynamical problem for all pertinent boundary conditions often encountered by practicing engineers.

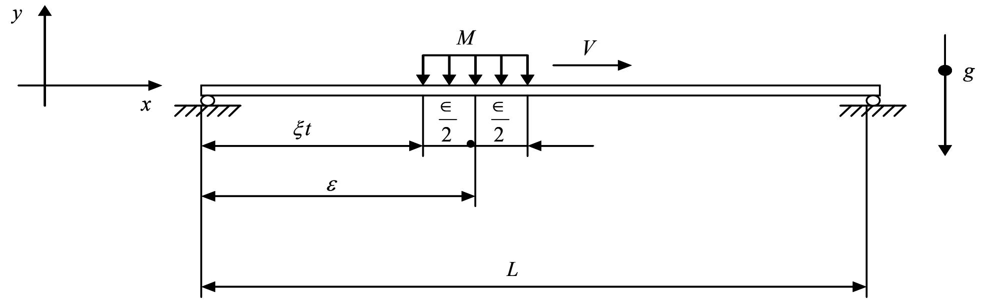

Figure 1: Schematic diagram of a beam under partially distributed load.(Esmailzadeh and Gorashi 1995).

2 THEORY AND FORMULATION OF THE PROBLEM

Consider the vibration of a structural member resting on elastic foundation and traversed by uniform partially distributed masses M. The mass M is assumed to strike the beam at the point x=0 and time t=0 and travels across it with a constant velocity v. The equation of motion, assuming uniform cross section is given by the fourth order partial differential equation.

$$EI\frac{\partial^4 z(x,t)}{\partial x^4} - N\frac{\partial^2 z(x,t)}{\partial x^2} + \mu\frac{\partial^2 z(x,t)}{\partial t^2} + Kz(x,t) - G\frac{\partial^2 z(x,t)}{\partial x^2} = Q(x,t) \tag{1}$$

where, E is the modulus of elasticity, I is the second moment of the beam's cross-section, μ is the mass per unit length of the beam, N is the axial force, K is the foundation constant, G is the shear rigidity, $z(x,t)$ is the deflection of the beam, measured upward from its equilibrium position when unloaded and $Q(x,t)$ is the travelling distributed load.

It is remarked here that the beam under consideration is assumed to have simple ends at both ends x=0 and x=L . Thus the boundary conditions are

$$z(0,t) = 0 = z(L,t), \frac{\partial^2 z(0,t)}{\partial x^2} = \frac{\partial^2 z(L,t)}{\partial x^2} \tag{2}$$

and the initial conditions of the motions of the slender member is given as

$$z(x,0) = 0 = \frac{\partial z(x,0)}{\partial t} \tag{3}$$

If the load inertia is taking into consideration, $Q(x,t)$ can be expressed as

$$Q(x,t) = Q_f(x,t)\left[1 - \frac{\Delta[z(x,t)]}{g}\right] \tag{4}$$

while the moving force $Q_f(x,t)$ acting on this engineering structure is given as

$$Q_f(x,t) = \frac{Mg}{\xi}\left[H\left(x-\gamma+\frac{\xi}{2}\right) - H\left(x-\gamma-\frac{\xi}{2}\right)\right] \tag{5}$$

where H is the Heaviside unit step function with the property,

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \tag{6}$$

and

$$\gamma = vt + \frac{\xi}{2} \tag{7}$$

For the limiting condition, as $\xi \to 0$ one obtains

$$\delta(x-v) = \frac{1}{\xi}\left[H\left(x-\gamma+\frac{\xi}{2}\right) - H\left(x-\gamma-\frac{\xi}{2}\right)\right] \tag{8}$$

where $\delta(x-v)$ is the Dirac delta function.

Furthermore, the operator Δ used in (4) is defined as

$$\Delta[\cdot] = \left(\frac{\partial^2}{\partial t^2} + 2\dot{\gamma}\frac{\partial^2}{\partial x \partial t} + \dot{\gamma}^2\frac{\partial^2}{\partial x^2} + \ddot{\gamma}\frac{\partial}{\partial x}\right)[\cdot] \tag{9}$$

Considering equation (5) and (8) would lead to the foundation for moving point mass. However, in this work ξ is not limited to be a small length.

Substituting (4) into (1) and taking into account (5) and (9) after some rearrangements gives

$$\begin{aligned} &EI\frac{\partial^4 z(x,t)}{\partial x^4} - N\frac{\partial^2 z(x,t)}{\partial x^2} + \mu\frac{\partial^2 Z(x,t)}{\partial t^2} + KZ(x,t) - G\frac{\partial^2 Z(x,t)}{\partial x^2} + \frac{M}{\xi}\left\{H\left(x-\gamma+\frac{\xi}{2}\right)\right. \\ &\left. - H\left(x-\gamma-\frac{\xi}{2}\right)\right\}\left[\frac{\partial^2 z(x,t)}{\partial t^2} + 2\xi\frac{\partial^2 Z(x,t)}{\partial x \partial t} + \xi^2\frac{\partial^2 Z(x,t)}{\partial x^2}\right] = \frac{Mg}{\xi}\left[H\left(x-\gamma+\frac{\xi}{2}\right) - H\left(x-\gamma-\frac{\xi}{2}\right)\right] \end{aligned} \tag{10}$$

which is the fourth order partial differential equation describing the flexural motions of our engineering member when traversed by uniform partially distributed masses. In what follows, an analytical technique that will be used to treat the problem above will be discussed.

3 SOLUTION PROCEDURES

To solve the problem above, a versatile method often used in problems involving mechanical vibrations will be adopted. This method will be employed to remove the ties in the governing differential equation (10) and to reduce it to sequence of second order ordinary differential equation with variable coefficients. This solution technique involves solving equation of the form.

$$\Pi[z(x,t)] - Q(x,t) = 0 \tag{11}$$

where

Π = The differential operator (linear or non-linear)

$z(x,t)$ = The structural displacement

$Q(x,t)$ = The transverse load acting on the structure

A sequence of linearly independent functions $U_i(x)$ which are the normalized deflection curves for the i$^{\text{th}}$ mode of the vibrating beam satisfying simply supported boundary conditions are chosen as

$$U_i(x) = \sin\frac{i\pi x}{L} \tag{12}$$

and appropriate solutions sought in the form

$$Z_i(x,t) = \sqrt{\frac{2}{L}}\sum_{i=1}^{\infty} Y_i(t)U_i(x) \tag{13}$$

The functions $Y_i(t)$ are the unknown functions of time to be determined. Thus, the unknown functions $Y_i(t)$ are obtained from the condition that the expression $U_i(x)$ should be orthogonal to the functions $U_j(x)$. In this way, we get a set of coupled ordinary differential equations

$$\int_0^L \left\{\Pi\left[\sum_{i=1}^{\infty} Y_i(t)U_i(x)\right] - Q(x,t)\right\}U_j(x)dx = 0 \tag{14}$$

From which we obtain $Y_i(t)$. These set of coupled second order ordinary differential equations are called Galerkin's equations. The set of coupled second order ordinary differential equations (14) are further treated using the modified asymptotic method of Struble. Furthermore, the following property of Heaviside function

$$\frac{1}{\xi}\int_0^L f(x)\left[H\left(x-\gamma+\frac{\xi}{2}\right) - H\left(x-\gamma-\frac{\xi}{2}\right)\right]dx = f(\xi) + \left(\frac{\xi}{2}\right)^2\frac{f''(\xi)}{3!} + \left(\frac{\xi}{2}\right)^4\frac{f^{iv}(\xi)}{5!} + \ldots \tag{15}$$

will be useful and also the function $Q(x,,t)$ is assumed to be expressible as

$$Q(x,t)=\sum_{i=1}^{\infty}U_i(x)\Omega_i(t) \tag{16}$$

where $\Omega_i(t)$ are unknown functions of time.

3.1 Operational Simplification

It is evident that an exact closed form solution of the partial differential equation (10) is impossible. Thus, substituting the expressions (13) and (15) and (16) into (10) after some rigorous mathematical procedures and rearrangements one obtains

$$\begin{aligned}
&\sum_{i=1}^{\infty}\Bigg[EI\sqrt{\frac{2}{L}}\left(\frac{i\pi}{L}\right)^4 Sin\frac{i\pi x}{L}Y_i(t)+N\sqrt{\frac{2}{L}}\left(\frac{i\pi}{L}\right)^2 Sin\frac{i\pi x}{L}Y_i(t)+\mu\sqrt{\frac{2}{L}}Sin\frac{i\pi x}{L}\ddot{Y}_i(t)+K\sqrt{\frac{2}{L}}Sin\frac{i\pi x}{L}Y_i(t)\\
&+G\sqrt{\frac{2}{L}}\left(\frac{i\pi}{L}\right)^2 Sin\frac{i\pi x}{L}Y_i(t)+\frac{M}{\xi}\sqrt{\frac{2}{L}}\left\{H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right\}Sin\frac{i\pi x}{L}\ddot{Y}_i(t)\\
&+\frac{2Mv}{\xi}\sqrt{\frac{2}{L}}\left(\frac{i\pi}{L}\right)\left\{H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right\}Cos\frac{i\pi x}{L}\dot{Y}(t)\\
&-\frac{Mv^2}{\xi}\sqrt{\frac{2}{L}}\left(\frac{i\pi}{L}\right)^2\left\{H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right\}Sin\frac{i\pi x}{L}Y_i(t)\Bigg]+\frac{Mg}{\xi}\sqrt{\frac{2}{L}}\left(\frac{i\pi}{L}\right)\left\{H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right\}=0
\end{aligned} \tag{17}$$

In order to determine an expression for $Y_i(t)$, it is required that the expression on the left hand side of equation (17) be orthogonal to the function $U_j(x)$. To this effect, multiplying equation (17) by $U_j(x)$, integrating from end x = 0 to end x=L and after some simplifications and rearrangements one obtains

$$\begin{aligned}
&\sum_{i=1}^{\infty}\Bigg\{\ddot{Y}_i(t)+\alpha_{mf}^2Y_i(t)+2\Gamma_\circ\Bigg\{Sin\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+\frac{\xi^2}{48}\left(\left(\frac{(i+j)\pi}{L}\right)^2Cos\frac{(i+j)\pi\gamma}{L}-\left(\frac{(i-j)\pi}{L}\right)^2Cos\frac{(i-j)\pi\gamma}{L}\right)\Bigg)\ddot{Y}_i(t)\\
&+2v\left(\frac{i\pi}{L}\right)\left(Cos\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+\frac{\xi^2}{48}\left(\left(\frac{(i+j)\pi}{L}\right)^2Cos\frac{(i+j)\pi\gamma}{L}-\left(\frac{(i-j)\pi}{L}\right)^2Cos\frac{(i-j)\pi\gamma}{L}\right)\right)\Bigg)\dot{Y}_j(t)\\
&-v^2\left(\frac{i\pi}{L}\right)^2Sin\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+\frac{\xi^2}{48}\left(\left(\frac{(i+j)\pi}{L}\right)^2Cos\frac{(i+j)\pi\gamma}{L}-\left(\frac{(i-j)\pi}{L}\right)^2Cos\frac{(i-j)\pi\gamma}{L}\right)\Bigg)Y_i(t)\Bigg\}=P_\circ Sin\frac{j\pi\gamma}{L}
\end{aligned} \tag{18}$$

where

$$\alpha_{mf}^2=\frac{EI}{\mu}\left(\frac{i\pi}{L}\right)^4+\frac{N}{\mu}\left(\frac{i\pi}{L}\right)^2+\frac{K}{\mu}+\frac{G}{\mu}\left(\frac{i\pi}{L}\right)^2,\quad P_\circ=\frac{Mg}{\mu}\sqrt{\frac{2}{L}}\left(1-\frac{1}{24}\left[\frac{j\pi\xi}{L}\right]^2\right) \tag{19}$$

Equation (18) is the transformed equation governing the problem of the uniform Euller-Bernoulli beam resting on the elastic sub-grade and subjected to uniform partially distributed parameter system. Now, considering the ith particle of the dynamical system, leads to

$$\begin{aligned}&\ddot{Y}_i(t)+\alpha_{mf}^2 Y_i(t)+2\Gamma_\circ\left\{Sin\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+\frac{\xi^2}{48}\left(\left(\frac{(i+j)\pi}{L}\right)^2 Cos\frac{(i+j)\pi\gamma}{L}-\left(\frac{(i-j)\pi}{L}\right)^2 Cos\frac{(i-j)\pi\gamma}{L}\right)\right)\ddot{Y}_i(t)\\&+2v\left(\frac{i\pi}{L}\right)\left(Cos\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+\frac{\xi^2}{48}\left(\left(\frac{(i+j)\pi}{L}\right)^2 Cos\frac{(i+j)\pi\gamma}{L}-\left(\frac{(i-j)\pi}{L}\right)^2 Cos\frac{(i-j)\pi\gamma}{L}\right)\right)\right)\dot{Y}_j(t)\\&-v^2\left(\frac{i\pi}{L}\right)^2 Sin\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+\frac{\xi^2}{48}\left(\left(\frac{(i+j)\pi}{L}\right)^2 Cos\frac{(i+j)\pi\gamma}{L}-\left(\frac{(i-j)\pi}{L}\right)^2 Cos\frac{(i-j)\pi\gamma}{L}\right)\right)\right\}=P_\circ Sin\frac{j\pi\gamma}{L}\end{aligned}\tag{20}$$

where

$$\Gamma_\circ=\frac{M}{\mu L}\tag{21}$$

Equation (20) is the transformed equation governing the motion of a uniform simply supported beam resting on bi-parametric elastic foundation and subjected to travelling masses. In what follows, a closed form solution of equation (20) is sought, to this end, we shall consider two special cases of equation (20) namely *the moving force* and *moving mass* problems.

3.2 The Moving Force Beam-Load Interaction Problem

The second order ordinary differential equation describing the behaviour of a thin beam resting on elastic sub-grade and under the actions of a uniform partially distributed moving force may be obtained from equation (20) by setting $\Gamma_\circ=0$. In this case, one obtains

$$\ddot{Y}_i(t)+\alpha_{mf}^2 Y_i(t)=P_\circ Sin\frac{j\pi\gamma}{L}\tag{22}$$

In view of (7) equation (22) can further be written as,

$$\ddot{Y}_i(t)+\alpha_{mf}^2 Y_i(t)=P_o Sin\frac{j\pi\left(vt+\frac{\xi}{2}\right)}{L}\tag{23}$$

Equation (23) is a classical case of a moving force problem associated with the system. Equation (23) after some simplifications yields

$$\ddot{Y}_i(t)+\alpha_{mf}^2 Y_i(t)=P_1 Sin\frac{j\pi vt}{L}+P_2 Cos\frac{j\pi vt}{L}\tag{24}$$

where

$$P_1=P_o Cos\frac{j\pi\xi}{2L}\ ,\ P_2=P_o Sin\frac{j\pi\xi}{2L}\tag{25}$$

To obtain an expression for $Y_i(t)$, equation (24) is subjected to a Laplace transformation defined as

$$(\tilde{\cdot}) = \int_0^\infty (\cdot) e^{-st} dt \tag{26}$$

where s is the Laplace parameter. Applying the initial conditions (3), one obtains the following algebraic equation

$$s^2 Y_i(s) + \alpha_{mf}^2 Y_i(s) = P_1 \frac{a}{s^2 + a^2} + P_2 \frac{s}{s^2 + a^2} \tag{27}$$

where

$$a = \frac{j\pi v}{L} \tag{28}$$

Equation (27) after some simplifications leads to

$$Y_i(s) = \frac{P_1}{\alpha_{mf}} \cdot \frac{\alpha_{mf}}{s^2 + \alpha_{mf}^2} \cdot \frac{a}{s^2 + a^2} + \frac{P_2}{\alpha_{mf}} \cdot \frac{\alpha_{mf}}{s^2 + \alpha_{mf}^2} \cdot \frac{s}{s^2 + a^2} \tag{29}$$

Thus, this problem reduces to that of finding the Laplace inversion of the equation (29) bove. To this effect, the following representations are adopted.

$$F(s) = \frac{\alpha_{mf}}{s^2 + \alpha_{mf}^2}, \quad G_1(s) = \frac{a}{s^2 + a^2} \quad \text{and} \quad G_2(s) = \frac{s}{s^2 + a^2} \tag{30}$$

so that the Laplace inversion of equation (29) is the convolution of F_i and G_i's defined as

$$F * G_i = \int_0^t F(t-u) G_i(u), \quad i = 1, 2 \tag{31}$$

Thus, the Laplace inversion of equation (29) is given as

$$Y_i(t) = \frac{P_1}{\alpha_{mf}} \cdot I_A + \frac{P_2}{\alpha_{mf}} \cdot I_B \tag{32}$$

Where

$$I_A = \int_0^t \left[\sin\alpha_{mf} t \cos\alpha_{mf} u - \cos\alpha_{mf} t \sin\alpha_{mf} u\right] \sin au du \tag{33}$$

$$I_B = \int_0^t \left[\sin\alpha_{mf} t \cos\alpha_{mf} u - \cos\alpha_{mf} t \sin\alpha_{mf} u\right] \cos au du \tag{34}$$

Evaluating integrals (33) and (34) one obtains

$$I_A = \frac{1}{\left(\alpha_{mf}^2 - a^2\right)}\left[\alpha_{mf} \sin at - a \sin\alpha_{mf} t\right] \tag{35}$$

$$I_B = \frac{1}{\left(\alpha_{mf}^2 - a^2\right)}\left[\alpha_{mf} \cos at - \alpha_{mf} \cos\alpha_{mf} t\right] \tag{36}$$

Substituting (35) and (36) into equation (32), gives the expression for $Y_i(t)$ as

$$Y_i(t) = \frac{1}{\alpha_{mf}\left(\alpha_{mf}^2 - a^2\right)}\left[P_1\left(\alpha_{mf} \sin at - a \sin\alpha_{mf} t\right) + P_2\left(\alpha_{mf} \cos at - \alpha_{mf} \cos\alpha_{mf} t\right)\right] \tag{37}$$

Thus, in view of (13), taking into account (12) and (37) gives

$$z(x,t) = \sqrt{\frac{2}{L}}\sum_{i=1}^{\infty}\frac{1}{\alpha_{mf}\left(\alpha_{mf}^2 - a^2\right)}\left[P_1\left(\alpha_{mf} \sin at - a \sin\alpha_{mf} t\right) + P_2\left(\alpha_{mf} \cos at - \alpha_{mf} \cos\alpha_{mf} t\right)\right]\cdot \sin\frac{i\pi x}{L} \tag{38}$$

which represents the transverse response of Euler-Bernoulli beam resting on elastic sub-grade and subject to uniform partially distributed moving forces when the inertial effect of the system is neglected.

3.3 The Moving Mass Beam-Load Interaction Problem

Since the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus, $\Gamma_0 \neq 0$ and one is required to solve the entire equation (20) when no term of the coupled differential equation is neglected. This is termed the moving mass problem. Unlike in the case of the moving force, an exact analytical solution to this equation is not possible. Thus, one resorts to an approximate analytical technique due to Struble discussed in [Gbadeyan and Oni (1992), Oni and Omolofe (2005b)]. By this technique, we seek the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the mass of the load. To this end, equation (20) is rearranged to take the form.

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\left\{ \ddot{Y}_i(t)+\left[\frac{\Gamma_\circ(H_2 Cos\frac{i\pi\gamma}{L}Sin\frac{i\pi\gamma}{L})}{1+\Gamma_\circ(2Sin^2\frac{i\pi\gamma}{L}+H_1Cos\frac{2i\pi\gamma}{L})}\right]\dot{Y}_i(t)\right.\\
&\left.+\left[\frac{\alpha_{mf}^2-\Gamma_\circ(H_4Sin^2\frac{i\pi\gamma}{L}H_5Cos\frac{2i\pi\gamma}{L})}{1+\Gamma_\circ(2Sin^2\frac{i\pi\gamma}{L}+H_1Cos\frac{2i\pi\gamma}{L})}\right]Y_i(t)\right\}\\
&+\sum_{\substack{i=1\\ i*j}}^{\infty}\left\{\left[\frac{\Gamma_\circ(2Sin\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+H_6Cos\frac{(i+j)\pi\gamma}{L}-H_7Cos\frac{(i-j)\pi\gamma}{L})}{1+\Gamma_\circ(2Sin^2\frac{i\pi\gamma}{L}+H_1Cos\frac{2i\pi\gamma}{L})}\right]\ddot{Y}_j(t)\right.\\
&+\left[\frac{\Gamma_\circ(H_2Cos\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}+H_8Cos\frac{(i+j)\pi\gamma}{L}-H_9Cos\frac{(i-j)\pi\gamma}{L})}{1+\Gamma_\circ(2Sin^2\frac{i\pi\gamma}{L}+H_1Cos\frac{2i\pi\gamma}{L})}\right]\dot{Y}_j(t)\\
&\left.-\left[\frac{\Gamma_\circ(H_{10}Sin\frac{i\pi\gamma}{L}Sin\frac{j\pi\gamma}{L}-H_{11}Cos\frac{(i+j)\pi\gamma}{L}+H_{12}Cos\frac{(i-j)\pi\gamma}{L})}{1+\Gamma_\circ(2Sin^2\frac{i\pi\gamma}{L}+H_1Cos\frac{2i\pi\gamma}{L})}\right]Y_j(t)\right\}\\
&=\left[\frac{\Gamma_\circ LgQ_\circ Sin\frac{j\pi\gamma}{L}}{1+\Gamma_\circ(2Sin^2\frac{i\pi\gamma}{L}+H_1Cos\frac{2i\pi\gamma}{L})}\right]
\end{aligned}
\tag{39}
$$

Where

$$
\begin{aligned}
&H_1=\frac{1}{6}\left(\frac{i\pi\xi}{L}\right)^2 \quad , \quad H_2=4v\left(\frac{i\pi}{L}\right) \quad , \quad H_3=\frac{\xi^2}{3}\left(\frac{i\pi}{L}\right)^3 \quad , \quad H_4=2\left(\frac{i\pi\xi}{L}\right)^2\\
&H_5=\frac{\xi^2}{6}\left(\frac{i\pi}{L}\right)^4 \quad , \quad H_6=\frac{1}{24}\left(\frac{(i+j)\pi\xi}{L}\right)^2 \quad , \quad H_7=\frac{1}{24}\left(\frac{(i-j)\pi\xi}{L}\right)^2 \quad , \quad H_8=\frac{1}{12}\left(\frac{i\pi}{L}\right)\left(\frac{(i+j)\pi\xi}{L}\right)^2\\
&H_9=\frac{v}{12}\left(\frac{i\pi}{L}\right)\left(\frac{(i-j)\pi\xi}{L}\right)^2 \quad , \quad H_{10}=2\left(\frac{i\pi v}{L}\right)^2 \quad , \quad H_{11}=\frac{1}{24}\left(\left(\frac{i\pi\xi}{L}\right)\left(\frac{(i+j)\pi}{L}\right)\right)^2 \quad , \quad H_{12}=\frac{1}{24}\left(\left(\frac{i\pi v}{L}\right)\left(\frac{(i-j)\pi\xi}{L}\right)\right)^2
\end{aligned}
\tag{40}
$$

Evidently, unlike the moving force problem, an exact analytical solution to equation (39) does not exist. Thus, a modification of the asymptotic method due to struble often used in treating weakly homogenous and non-homogenous non-linear oscillatory system is resorted to. By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the moving mass. Following the procedures extensively discussed in [Oni and Omolofe (2005b)], the homogeneous part of equation (39) is simplified to take the form

$$\frac{d^2 Y_i(t)}{dt^2} + \alpha_{mm}^2 Y_i(t) = 0 \tag{41}$$

where

$$\alpha_{mm} = \alpha_{mf}\left[1 - \Gamma_*\left((2 - \frac{1}{2}H_1) + \frac{(H_4 - \frac{H_5}{2})}{\alpha_{mf}^2}\right)\right] \tag{42}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Thus, the entire equation (39) reduces to

$$\frac{d^2 Y_i(t)}{dt^2} + \alpha_{mm}^2 Y_i(t) = \Gamma_* L g Q_o \sin\frac{j\pi\left(vt + \frac{\xi}{2}\right)}{L} \tag{43}$$

which when solved in conjunction with the initial conditions yields an expression for $Y_i\left(t\right)$ as

$$Y_i(t) = \frac{1}{\alpha_{mm}\left(\alpha_{mm}^2 - a^2\right)}\left[Q_1\left(\alpha_{mm}\sin at - a\sin\alpha_{mm}t\right) + Q_2\left(\alpha_{mm}\cos at - \alpha_{mm}\cos\alpha_{mm}t\right)\right] \tag{44}$$

where

$$Q_1 = \Gamma^* L g Q_o \cos\frac{j\pi\xi}{2L}\ ,\ Q_2 = \Gamma^* L g Q_o \sin\frac{j\pi\xi}{2L} \tag{45}$$

Thus, in view of (13), taking into account (12) and (44) gives

$$z(x,t) = \sqrt{\frac{2}{L}}\sum_{i=1}^{\infty}\frac{1}{\alpha_{mm}\left(\alpha_{mm}^2 - a^2\right)}\left[Q_1\left(\alpha_{mm}\sin at - a\sin\alpha_{mm}t\right) + Q_2\left(\alpha_{mm}\cos at - \alpha_{mm}\cos\alpha_{mm}t\right)\right]\cdot\sin\frac{i\pi x}{L} \tag{46}$$

Equation (46) represents the transverse response of a simply supported Euler-Bernoulli beam resting on an elastic sub-grade and subject to uniform partially distributed moving mass.

4 COMMENTS ON THE CLOSED FORM SOLUTIONS

In an important study such as this, investigating the phenomenon of resonance is very crucial because the transverse displacement of an elastic beam may increase without limit. It is seen from equation (38) that the simply supported beams on elastic subgrade and under the actions of travelling distributed forces reaches a state of resonance whenever

$$\alpha_{mf} = \frac{k\pi v}{L} \tag{47}$$

while equation (46) clearly shows that the same beam under the actions of moving distributed masses will experience resonance effects whenever

$$\alpha_{mm} = \frac{k\pi v}{L} \tag{48}$$

but, from equation (42)

$$\alpha_{mm} = \alpha_{mf}\left[1-\Gamma_0\left(\left(2-\frac{1}{2}H_1\right)+\frac{\left(H_4-\frac{1}{2}H_5\right)}{\alpha_{mf}^2}\right)\right] \tag{49}$$

which implies

$$\alpha_{mf} = \frac{\frac{k\pi v}{L}}{1-\Gamma_0\left(\left(2-\frac{1}{2}H_1\right)+\frac{\left(H_4-\frac{1}{2}H_5\right)}{\alpha_{mf}^2}\right)} \tag{50}$$

It is therefore clear that, for the same natural frequency, the critical speed for the system consisting of a simply supported Bernoulli-Euler beam resting on elastic foundation and under the actions of travelling distributed force is greater than that of moving distributed mass problem. Thus, for the same natural frequency, resonance is reached earlier in the moving distributed mass than in the moving distributed force system.

5 ANALYSIS OF RESULT AND DISCUSSION

In this section, the analysis proposed in the previous sections are illustrated by considering a homogenous beam of modulus of elasticity $E = 2.9012\times10^9 N/m^2$, the moment of inertial $I = 2.87698\times10^{-3} kgm^2$, the beam span $L = 12.192m$ and the mass per unit length of the beam $\mu = 2758.291 kg/m$. The load is also assumed to travel with constant velocity $V = 8.128 m/s$. The values of foundation moduli are varied between $0N/m^3$ and $40000\,N/m^3$, the values of axial force N varied between $0N$ and $2.0\times 10^8 N$.

Figure 2 displays the transverse displacement response of a simply supported uniform beam under the action of uniform partially distributed forces travelling at constant velocity for the various values of axial force N and for fixed values of subgrade moduli K=40000 and shear modulus G=30000. The figures show that as N increases, the response amplitude of the uniform beam decreases. Similar results are obtained when the simply supported beam is subjected to partially distributed mass travelling at constant velocity as shown in figure 8. For various travelling time t, the displacement response of the beam for various values of subgrade moduli K and for fixed values of axial forcé N=20000 and shear modulus G=30000 are shown in figure 3. It is observed that higher values of subgrade moduli K reduce the deflection of the vibrating beam. The same behavior characterizes the response of the simply supported beam under the actions of uniform partially distributed masses moving at constant velocity for various values of subgrade moduli K as shown in figure 9. Also, figures 4 and 10 display the deflection profile of the simply supported uniform beam respectively to partially distributed forces and masses travelling at constant velocity for various values of shear modulus and fixed values of axial forcé N=20000 and subgrade moduli K=40000. These figures clearly show that as the value of the shear moduli increases, the deflection of the simply supported uniform beam under the action of both moving forces and masses travelling at constant velocity decreases.

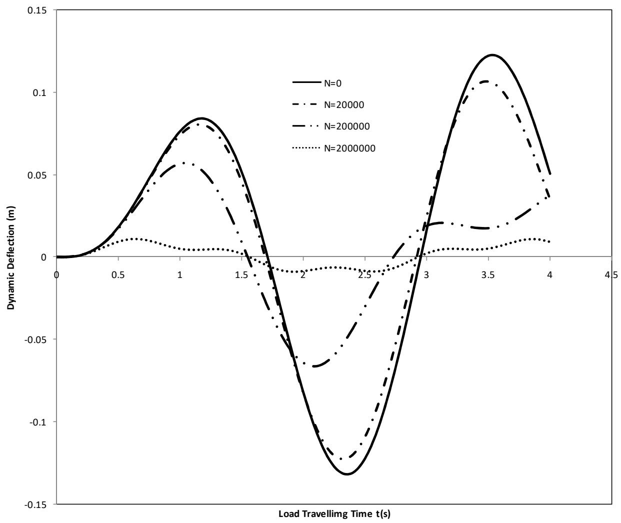

Figure 2: Transverse displacement response of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of axial force N and for fixed values of K = 40000, G = 30000.

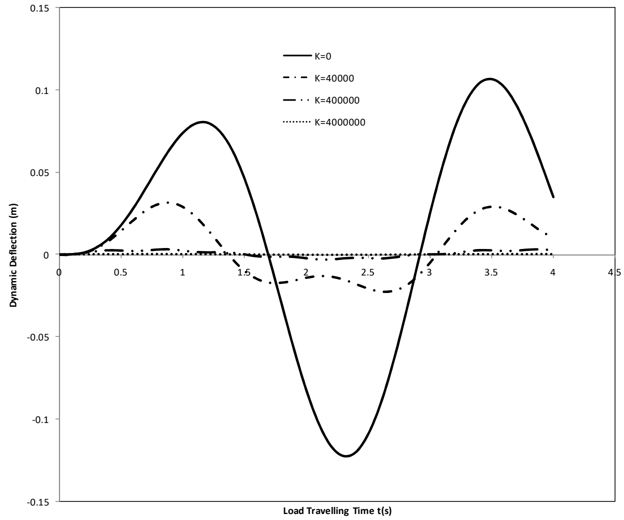

Figure 3: Displacement response of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of foundation modulus K and for fixed values of N = 20000, G = 30000.

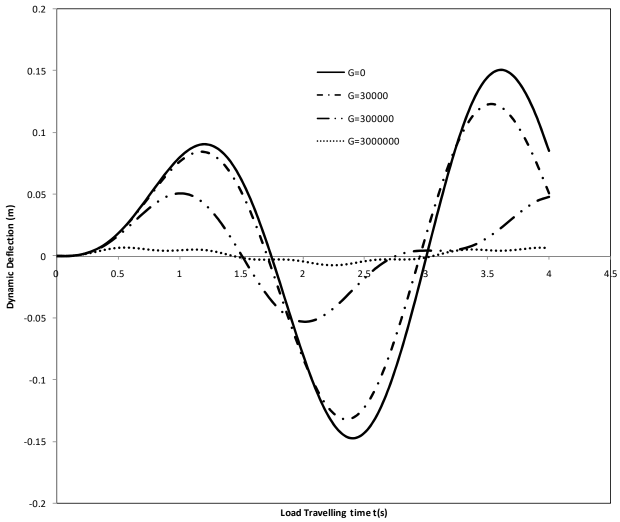

Figure 4: Deflection Profile of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of shear modulus G and for fixed values of K=40000 and N = 20000.

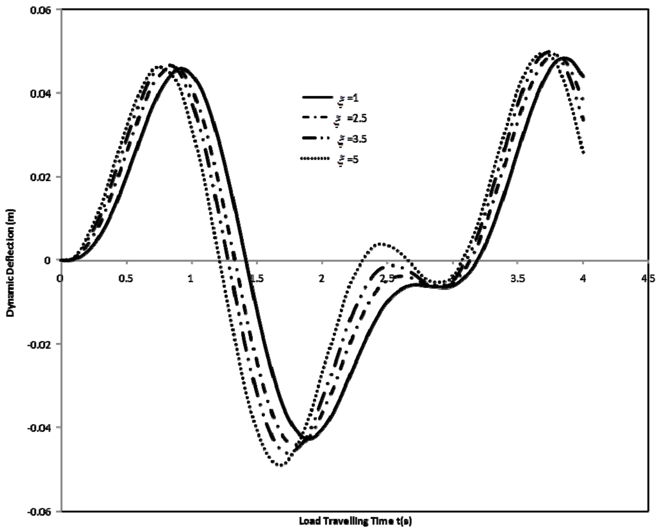

Figure 5: Response Amplitude of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of the load width and for fixed values of G = 30000, K=40000 and N = 20000.

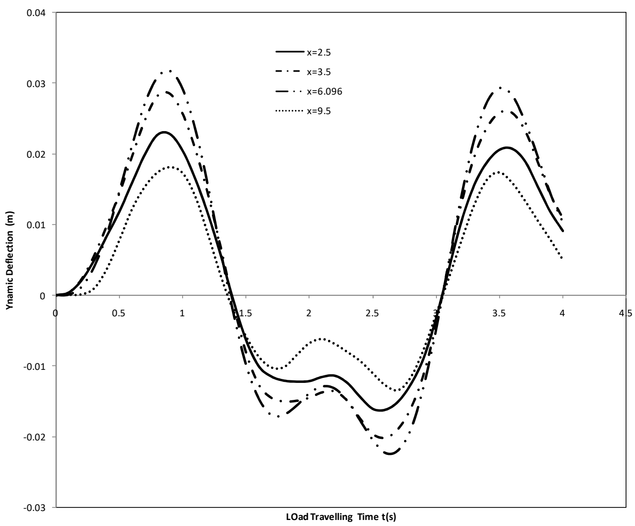

Figure 6: Response of a simply supported structural members resting on elastic foundation to uniform partially distributed forces for various values of the load position x and for fixed values of G = 30000, K=40000 and N = 20000.

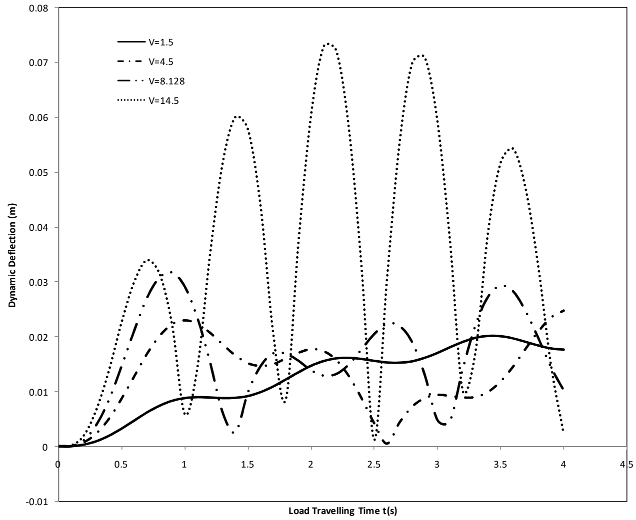

Figure 7: Response characteristics of a simply supported structural members resting on elastic foundation to uniform partially distributed forces for various values of the travelling load velocities and for fixed values of G = 30000, K=40000 and N = 20000.

Figure 5 displays the response amplitude of a simply supported uniform beam under the action of uniform partially distributed forces travelling at constant velocity for various values of load width ξ and for fixed values of subgrade moduli K=40000, axial force N=20000 and shear modulus G=30000. The figure show that as the width increases, the effects of the width on the response amplitude of the uniform beam increases as the load progresses on the structure. Similar results are obtained when the simply supported beam is subjected to partially distributed masses travelling at constant velocity as shown in figure 11. For various travelling time t, the response of the beam for various values of travelling load positions x and for fixed values of axial forcé N=20000, subgrade modulus K=40000 and shear modulus G=30000 are shown in figure 6. It is observed that the impact of the travelling load is greatest at the middle of this vibrating solid structure. The same behavior characterizes the response of the simply supported beam under the actions of uniform partially distributed masses moving at constant velocity for different travelling load positions as shown in figure 12.

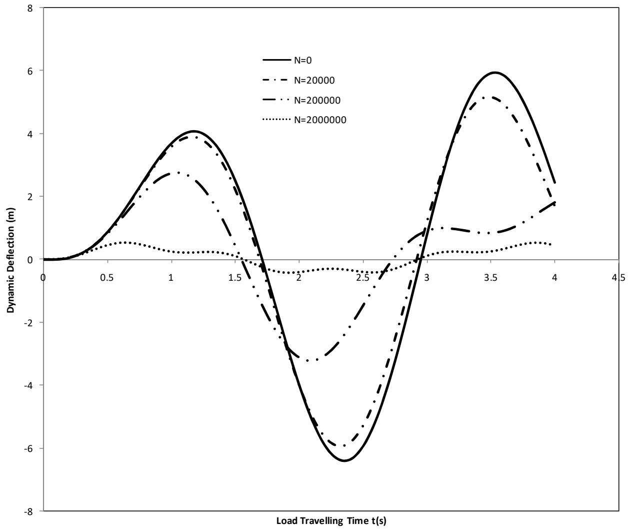

Figure 8: Transverse displacement response of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of axial force N and for fixed values of K = 40000, G = 30000.

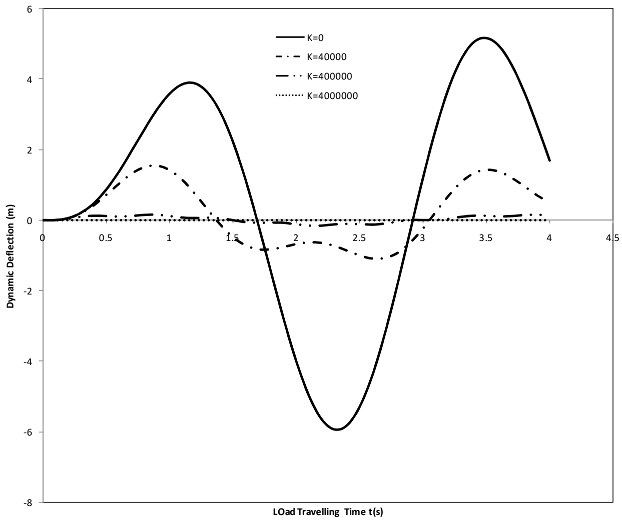

Figure 9: Displacement response of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of foundation modulus K and for fixed values of N = 20000, G = 30000.

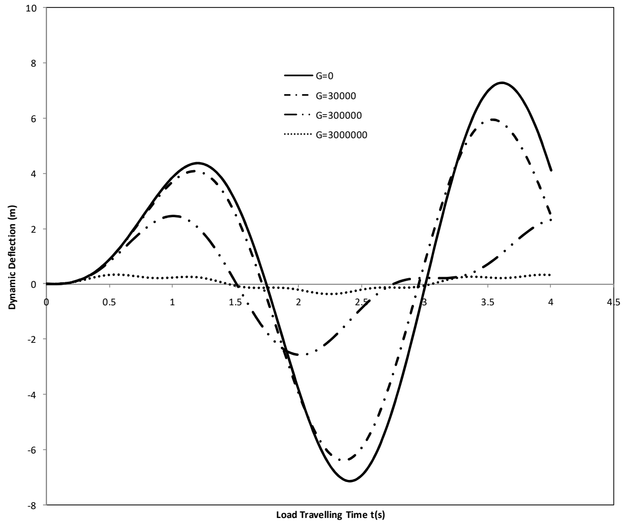

Figure 10: Deflection Profile of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of shear modulus G and for fixed values of K=40000 and N = 20000.

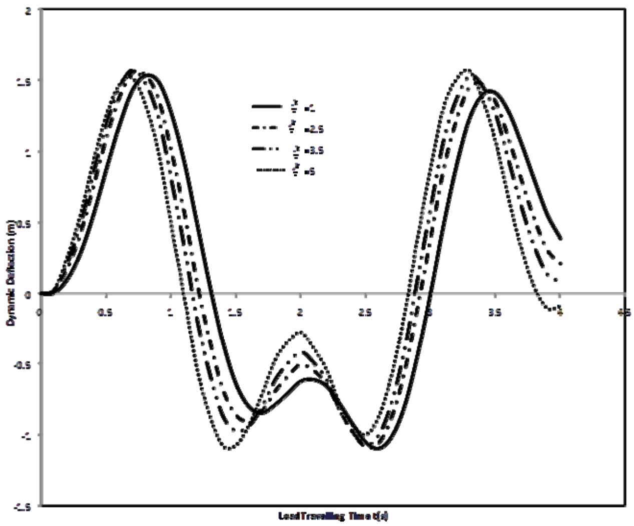

Figure 11: Response Amplitude of a simply supported structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of the load width and for fixed values of G = 30000, K=40000 and N = 20000.

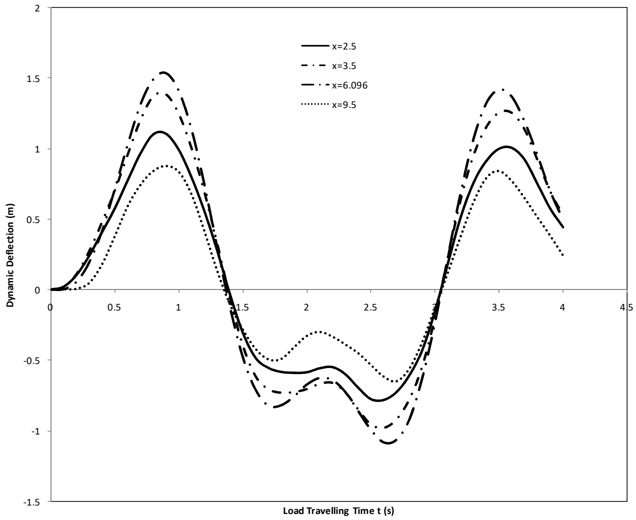

Figure 12: Response of a simply supported structural members resting on elastic foundation to uniform partially distributed masses for various values of the load position x and for fixed values of G = 30000, K=40000 and N = 20000.

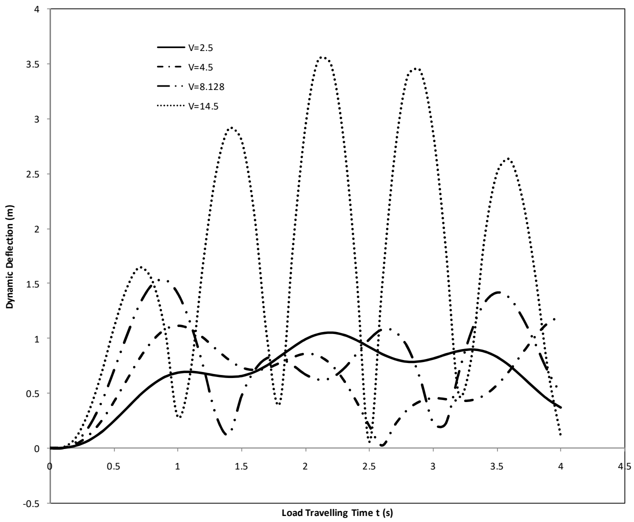

Figure 13: Response characteristics of a simply supported structural members resting on elastic foundation to uniform partially distributed masses for various values of the travelling load velocities and for fixed values of G = 30000, K=40000 and N = 20000.

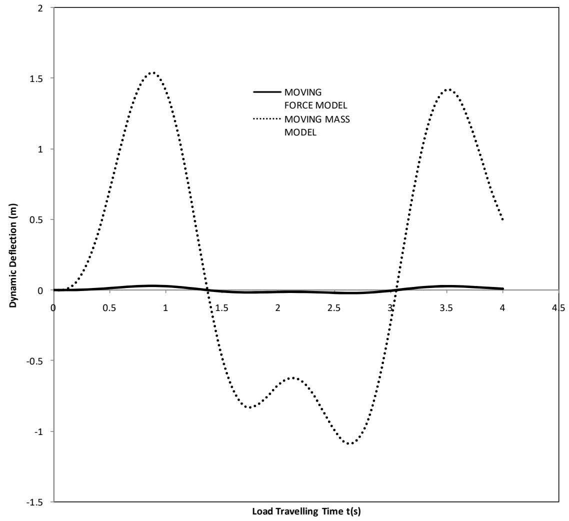

Figure 14: Comparison of the dynamic characteristic of moving force and moving mass cases of a uniform simply supported beam for fixed values of G = 0, K=0 and N =0.

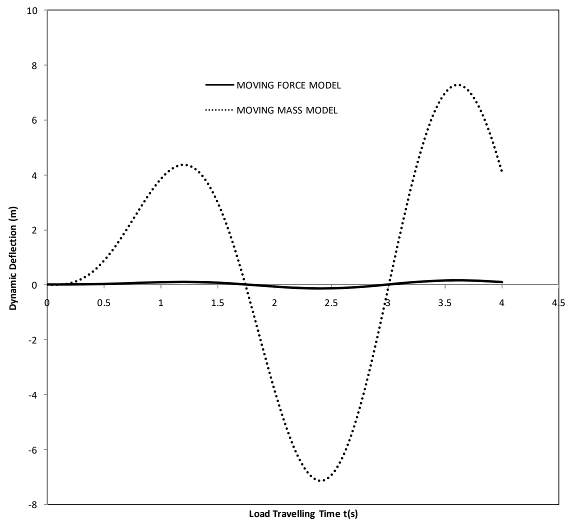

Figure 15: Comparison of the dynamic characteristic of moving force and moving mass cases of a uniform simply supported beam for fixed values of G = 30000, K=40000 and N = 20000.

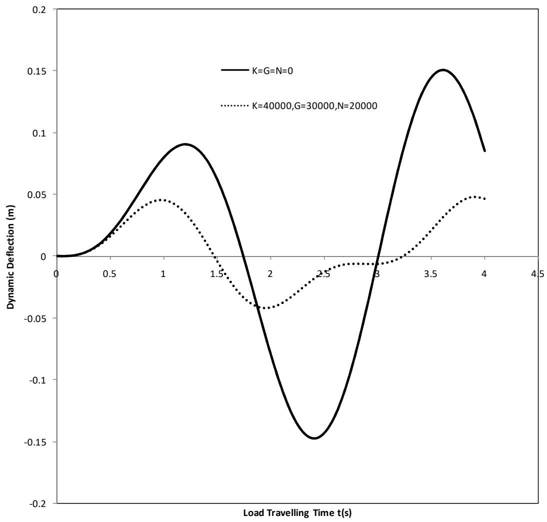

Figure 16: Comparison of the deflection profiles of the simply supported moving force uniform beam for values K=G=N=0 versus G = 30000, K=40000 and N = 20000.

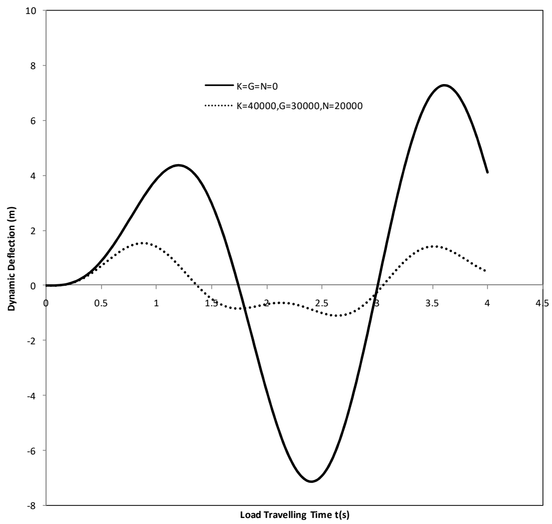

Figure 17: Comparison of the deflection profiles of the simply supported moving mass uniform beam for values K=G=N=0 versus G = 30000, K=40000 and N = 20000.

Figure 7 displays the response characteristics of a simply supported uniform beam under the action of uniform partially distributed forces for various values of load velocity V and for fixed values of subgrade moduli K=40000, axial force N = 20000 and shear modulus G = 30000. The figures show that the higher the velocity the larger the deflection of the vibrating structure. Similar results are obtained when the simply supported beam is subjected to partially distributed masses as displayed in figure 13. Figures 14 and 15 depict the comparison of the response characteristics of the moving force and moving mass cases of a simply supported uniform beam traversed by a moving distributed load travelling at constant velocity for fixed values of N=0, K=0, G=0 and N=20000, K=40000 and G=30000. From these figures, it is seen that the dynamic deflection of the beam under the actions of the moving load is greatly affected when the structural parameters N, K and G are incoporated into the governing equation of motion. Figure 16 compares the deflection profiles of the moving force model of the beam for the two set of values K=0, N=0, G=0 and N=20000, K=40000 and G=30000. It is deduced from this figure that the amplitude of deflection for the set of values N=0, K=0 and G=0 is much higher tan that of the set of values N=20000, K=40000 and G=30000. Similar result is obtained for a moving mass model of this structural member as shown in figure 17.

6 CONCLUDING REMARKS

The classical problem of the response characteristics of uniform structural member resting on elastic subgrade and subject to uniform partially distributed load is studied in this work. The closed form solutions of the governing fourth order partial differential equations with variable coefficients are presented for the moving force and mass models. Various results and analyses are carried out and phenomenon of resonance is studied for the dynamical system. The findings of this study exhibit among others the following useful and interesting features:

i. This study has provided a useful information on the effect of axial force, foundation stiffness, load width, load velocity and shear modulus on uniform Bernoulli-Euler beam resting on Pasternak foundation and under the action of uniform partially distributed loads.

ii. Results also show that higher values of the structural parameters K, N and G are required in the case of moving mass problem than that of the moving force beam problem.

iii. It is found that the dynamic stability of the elastic beam subjected to moving load is greatly enhanced with the presence of the structural parameters N, K and G.

iv. This study has also provided a very useful information on the conditions under which the vibrating system will experience the occurrence of undesirable phenomenon called resonance for both the moving distributed force and mass problems involving Bernoulli –Euler beams.

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