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A damage constitutive model accounting for induced anisotropy and bimodular elastic response

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Abstract

In several classes of materials progressive diffuse damage is responsible by severe changes of the mechanical response. Continuous Damage Mechanics (CDM) is a proper tool to formulate damage constitutive relationships including such kind of features. Following CDM, a constitutive model is proposed here by exploring the fundamental hypothesis of energy equivalence between real and continuous medium. According to the proposed modeling, the material is assumed as an initial elastic isotropic medium presenting anisotropy and permanent strains induced by damage evolution. Moreover, damage can also induce a bimodular response in the material, i.e., distinct elastic responses whether traction or compression stress states prevail. To conveniently take into account bimodularity, two damage tensors governing the rigidity in traction or compression regimes are introduced. A certain criteria are then proposed in order to characterize the dominant states. On the other hand, damage criteria indicating the initial and further evolution of damage are expressed in terms of strain energy densities. The model ability to reproduce basic experimental responses is illustrated by comparing some results varying from one to three-axial stress states. A frame structure behavior is then simulated in order to show the potentialities of the model employment to handle large problems.

Keywords: damage mechanics, constitutive model, anisotropy, bimodular materials

1 Introduction

Many fiber-reinforced composite materials exhibit intrinsic anisotropy and bimodularity, i.e., distinct responses in tension and compression prevailing states [4]. On the other hand, brittle materials such as concrete are a kind of composites that can be initially considered isotropic and unimodular. However when they have been damaged, those materials would start to present some degree of anisotropy and bimodularity [8]. Assuming small deformations, a formulation

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of constitutive laws for either initially isotropic or anisotropic elastic bimodular materials was proposed by [3]. Such proposition considers a bimodular hyperelastic material, being defined an elastic potential energy density which must be once continuously differentiable (whole wise), but only piecewise twice continuously differentiable. The stress-strain relationship derived from this potential is piecewise continuously differentiable leading to an elasticity tensor discontinuous referred to a hypersurface that contains the origin and divides the strain space into a compression and tension sub-domains. In this way, the modeling is able to produce different response in tension and compression.

In this work the formulation of [3] is extended to incorporate damage effects. In particular a constitutive model for concrete is derived. Accordingly, the material is initially considered as an isotropic continuous medium with anisotropy and bimodularity induced by the damage. On one side the class of anisotropy induced and considered in the model (transversal isotropy) elapses from the assumption that locally the loaded concrete always presents a diffuse oriented damage distribution as appointed by experimental observations [5, 9, 14]. On the other side, the bimodularity induced by damage is captured by the definition of two damage tensors: one for dominant tension states and another one for dominant compression states.

This paper is divided into six sections. In section 2, the extended framework of Curnier incorporating the damage effect is discussed. In section 3, it is presented the proposed constitutive model for the concrete and some aspects of its formulation are detailed. In the section 4 the good accuracy of the model is illustrated by comparing some numerical and experimental responses, from one to three-axial stress states. In the section 5 the model is then applied in one-dimensional analysis of a reinforced concrete frame. Finally, a few conclusions are discussed in section 6.

2 Extended formulation for anisotropic elastic media with damage and bimodular response

In this section the original proposal of [3] is extended to take into account the damage effects.

Accordingly with, the coefficients named bulk (λ_{ab}) and shear (μ_a) modulus are considered as functions of the damage state, so that the stress-strain relationship would be influenced by damage variables. Moreover, the hypersurface g ($\boldsymbol{\varepsilon}$, D_i) taken as the criterion for the identification of the constitutive responses in compression or traction would be also influenced by the damage variables.

A more specific aspect to be pointed out is related to the thermodynamically associated variables to the damage ones. Such variables can be interpreted as rates of energy released during the damage evolution process. Those associated variables can be used in the definition of a criterion to identify the beginning and evolution of the damage.

Then, the relation for the extended potential energy function valid for general cases of anisotropy can be written in the following form:

$$W(D_i, g(\varepsilon), \varepsilon) = \frac{\lambda_{ab}(D_i, g(\varepsilon))}{2} tr(\mathbf{A}_a \varepsilon) tr(\mathbf{A}_b \varepsilon) + \mu_a(D_i) tr(\mathbf{A}_a \varepsilon^2)(a, b = 1, d) \tag{1}$$

where the subscript i can assume values from 1 to the number of damage scalar variables considered by the model, while d = 1 for isotropy (recovering then the two usual Lamé constants), d = 2 for transverse isotropy (5 constants) and d = 3 for orthotropy (9 constants). The tensors A_a and A_b may be defined similarly as in [3]: (A_1 = I) for isotropic materials, (A_1 = I, A_2 = A) for transverse isotropic materials and (A_1 = I, A_2 = A, A_3 = B) for orthotropic materials.

Let us now consider that the the material behaves initially as an isotropic medium with the same stiffness in tension and compression. With the appearance and evolution of the damage, the material passes to present a bimodular and anisotropic behavior. Then, by assuming a generic situation in which the medium already presents a certain damage level, which has lead to a transverse isotropy state, the elastic energy potential and the derived relationships to stress and rigidity modulus are given by:

$$\begin{aligned} W(D_i, g(\varepsilon, D_i), \varepsilon) &= \tfrac{\lambda_{11}}{2} tr^2(\varepsilon) + \mu_1 tr(\varepsilon^2) - \tfrac{\lambda_{22}(D_i, g(\varepsilon, D_i))}{2} tr^2(\mathbf{A}\varepsilon) + \\ &\quad -\lambda_{12}(D_i, g(\varepsilon, D_i)) tr(\varepsilon) tr(\mathbf{A}\varepsilon) - \mu_2(D_i) tr(\mathbf{A}\varepsilon^2) \end{aligned} \tag{2}$$

$$\begin{aligned} \sigma(D_i, g(\varepsilon, D_i), \varepsilon) &= \lambda_{11} tr(\varepsilon)\mathbf{I} + 2\mu_1 \varepsilon - \lambda_{22}(D_i, g(\varepsilon, D_i)) tr(\mathbf{A}\varepsilon)\mathbf{A} + \\ &\quad -\lambda_{12}(D_i, g(\varepsilon, D_i))[tr(\varepsilon)\mathbf{A} + tr(\mathbf{A}\varepsilon)\mathbf{I}] - \mu_2(D_i)[\mathbf{A}\varepsilon + \varepsilon\mathbf{A}^T] \end{aligned} \tag{3}$$

$$\begin{aligned} \mathbf{E}(D_i, g(\varepsilon, D_i), \varepsilon) &= \mathbf{E}_0 - \lambda_{22}(D_i, g(\varepsilon, D_i))[\mathbf{A} \otimes \mathbf{A}] + \\ &\quad -\lambda_{12}(D_i, g(\varepsilon, D_i))[\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}] - \mu_2(D_i)[\mathbf{A} \underline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{A}] \end{aligned} \tag{4}$$

The variables associated to damage can be obtained from the energy function by taken the gradient in terms of the damage variables, as follows:

$$\begin{aligned} Y(D_i, g(\varepsilon, D_i), \varepsilon) &= \nabla_D W = \tfrac{\partial W}{\partial D_i} = -\tfrac{1}{2} \tfrac{\partial \lambda_{22}(D_i, g(\varepsilon, D_i))}{\partial D_i} tr^2(\mathbf{A}\varepsilon) + \\ &\quad - \tfrac{\partial \lambda_{12}(D_i, g(\varepsilon, D_i))}{\partial D_i} tr(\varepsilon) tr(\mathbf{A}\varepsilon) - \tfrac{\partial \mu_2(D_i)}{\partial D_i} tr(\mathbf{A}\varepsilon^2) \end{aligned} \tag{5}$$

where $\lambda_{11} = \lambda_0$ e $\mu_1 = \mu_0$ are the Lamé constants and E_0 is the initial isotropic elastic stiffness tensor.

The bimodular character is taken into account by the following conditions:

$$\begin{aligned} \lambda_{12}(D_i, g(\varepsilon, D_i)) &:= \begin{cases} \lambda_{12}^-(D_i) & \text{if} \quad g(\varepsilon, D_i) < 0 \\ \lambda_{12}^+(D_i) & \text{if} \quad g(\varepsilon, D_i) > 0 \end{cases} ; \\ \lambda_{22}(D_i, g(\varepsilon, D_i)) &:= \begin{cases} \lambda_{22}^-(D_i) & \text{if} \quad g(\varepsilon, D_i) < 0 \\ \lambda_{22}^+(D_i) & \text{if} \quad g(\varepsilon, D_i) > 0 \end{cases} \end{aligned} \tag{6}$$

It must be pointed out that the conditions above impose a jump in the elasticity tensor across the hypersurface, g ($\boldsymbol{\varepsilon}$, D_i). The jump has the normal direction to the hypersurface. Besides, the shear coefficients μ_a must be the same in tension and in compression, as explained in [3].

Actually, the choice for the number of damage variables and class of anisotropy depends of the material that will be modeled. In this context, the present formulation can be extended to include more complex level of anisotropy.

3 Constitutive model for the concrete

The concrete is here assumed as an initially isotropic material that starts to present transverse isotropy and bimodular responses induced by the damage. The model formulation is built from the formalism presented in the previous section. Moreover, the model tries to respect the principle of energy equivalence between damaged real medium and equivalent continuous medium established in the CDM [6]. Thus, the rigidity and flexibility constitutive tensors of the equivalent continuous medium result symmetric.

Note that general forms to the fourth-order damage tensor D can be proposed in order to take into account the anisotropy induced by damage. In this work the definition of that tensor follows a so-called scalar form expressed as: $D = f_j(D_i)\ M_j$, where $f_j(D_i)$ are scalar valued functions of the damage scalar variables D_i and M_j are anisotropic tensors. In the case of this model, the particular adopted tensors to M_j are the ones that allow representing the transversal isotropy.

Concerning bimodularity induced by damage, it is interesting to define two damage tensors, one for dominant tension stress states and another one for dominant compression stress states.

To the dominant tension states, the following scalar damage tensor is here proposed:

$$\mathbf{D}_T = f_1(D_1, D_4, D_5)(A \otimes A) + 2f_2(D_4, D_5)[(A \underline{\overline{\otimes}} I + I \underline{\overline{\otimes}} A) - (A \otimes A)] \tag{7}$$

where $f_1(D_1, D_4, D_5) = D_1 - 2\ f_2(D_4, D_5)$ and $f_2(D_4, D_5) = 1 - (1\text{-}D_4)\ (1\text{-}D_5)$.

The variable D_1 represents the damage in direction orthogonal to the transverse isotropy local plane of the material, while D_4 is representative of the damage generated by the sliding movement between the crack faces. The third damage variable, D_5, is only activated if a previous compression state accompanied by damage has occurred.

In the Equation (7), the tensor I is the second-order identity tensor and the tensor A, by definition, [3], is formed by dyadic product of the unit vector perpendicular to the transverse isotropy plane for himself.

For dominant compression states, it is proposed the following relationship for the damage tensor:

$$\mathbf{D}_C = f_1^*(D_2, D_4, D_5)(A \otimes A) + f_2(D_3)[(I \overline{\otimes} I) - (A \otimes A)] + 2f_3(D_4, D_5)[(A \underline{\overline{\otimes}} I + I \underline{\overline{\otimes}} A) - (A \otimes A)] \tag{8}$$

being $f_1^*(D_2, D_4, D_5) = D_2 - 2\ f_3(D_4, D_5)$,$f_2(D_3) = D_3$ and $f_3(D_4, D_5) = 1 - (1\text{-}D_4)\ (1\text{-}D_5)$.

Note that the compression damage tensor introduces two additional scalar variables in its composition: D_2 and D_3. The variable D_2 (damage perpendicular to the transverse isotropy local plane of the material) reduces the Young's modulus in that direction and in conjunction to D_3 (that represents the damage in the transverse isotropy plane) degrades the Poisson's ratio throughout the perpendicular planes to the one of transverse isotropy.

It must be noted that the described forms for $f_j(D_i)$ are appropriate in the sense that they allow capturing the damage of the shear module as well as respect the hypothesis of tangential jump null of the constitutive tensor. If we consider a matrix representation and assuming, for instance, that the transversal isotropy local plane to be coincident with the 2-3 plane, both tensors D_T and D_C may be described as follows:

$$D_T = \begin{bmatrix} D_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-(1-D_4)(1-D_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-(1-D_4)(1-D_5) \end{bmatrix} ;$$

$$D_C = \begin{bmatrix} D_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-(1-D_4)(1-D_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-(1-D_4)(1-D_5) \end{bmatrix}$$

Furthermore it is possible to show that the resulting forms for D_T and D_C respect the energy equivalence principle, providing a symmetrical rigidity tensor. Finally, the resultant constitutive tensors are described by:

$$\begin{aligned} E_T \;=\;& \lambda_{11}[I \otimes I] + 2\mu_1[I \underline{\otimes} I] - \lambda_{22}^{+}(D_1, D_4, D_5)[A \otimes A] - \lambda_{12}^{+}(D_1)[\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}] + \\ & -\mu_2(D_4, D_5)[A \underline{\otimes} I + I \underline{\otimes} A] \end{aligned} \tag{9}$$

$$\begin{aligned} E_C \;=\;& \lambda_{11}[I \otimes I] + 2\mu_1[I \underline{\otimes} I] - \lambda_{22}^{-}(D_2, D_3, D_4, D_5)[A \otimes A] - \lambda_{12}^{-}(D_2, D_3)[\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}] + \\ & -\lambda_{11}^{-}(D_3)[\mathbf{I} \otimes \mathbf{I}] - \tfrac{(1-2\nu_0)}{\nu_0}\lambda_{11}^{-}(D_3)[\mathbf{I} \underline{\otimes} \mathbf{I}] - \mu_2(D_4, D_5) \quad [A \underline{\otimes} I + I \underline{\otimes} A] \end{aligned} \tag{10}$$

where $\lambda_{11} = \lambda_0$ and $\mu_1 = \mu_0$. The remaining parameters will only exist for no-null damage, evidencing in that way the anisotropy and bimodularity induced by damage. Those parameters are given by:

$$\lambda_{22}^{-}(D_2, D_3, D_4, D_5) = (\lambda_0 + 2\mu_0)(2D_2 - D_2^2) - 2\lambda_{12}^{-}(D_2, D_3) + \frac{(\nu_0 - 1)}{\nu_0}\lambda_{11}^{-}(D_3) - 2\mu_2(D_4, D_5)$$

$$\lambda_{12}^{-}(D_2, D_3) = \lambda_0[(1 - D_3)^2 - (1 - D_2)(1 - D_3)]$$

$$\lambda_{11}^{-}(D_3) = \lambda_0(2D_3 - D_3^2); \quad \mu_2(D_4, D_5) = 2\mu_0[1 - (1 - D_4)^2(1 - D_5)^2] \tag{11}$$
$$\lambda_{22}^{+}(D_1, D_4, D_5) = (\lambda_0 + 2\mu_0)(2D_1 - D_1^2) - 2\lambda_{12}^{+}(D_1) - 2\mu_2(D_4, D_5)$$
$$\lambda_{12}^{+}(D_1) = \lambda_0 D_1; \quad \mu_2(D_4, D_5) = 2\mu_0[1 - (1 - D_4)^2(1 - D_5)^2]$$

On a basis of a purely matricial interpretation, the different dyadic products appearing in Eqs.(9) and (10) have the function of allocating the material constants in certain positions of the rigidity tensors. Those products are discussed in the appendix.

3.1 Criterion for partition of strain space

In [3], it is defined a hypersurface in the stress or strain space to be used for the identification of the bimodularity constitutive response. In this work a particular form is adopted for the hypersurface in the strain space: a hyperplane g(ε,D) defined by the unit normal N (||N|| = 1) and characterized by its dependence of the strain and damage states.

To simplify the presentation, the hyperplane will be here expressed as the one obtained by enforcing the direction 1 in the strain space to be perpendicular to the transverse isotropy local plane. Then, referring to general cases of loading, the following relationship is proposed for the hyperplane:

$$g(\varepsilon, \mathbf{D}_T, \mathbf{D}_C) = \mathbf{N}(\mathbf{D}_T, \mathbf{D}_C).\varepsilon^e = \gamma_1(D_1, D_2) \quad \varepsilon_V^e + \gamma_2(D_1, D_2) \quad \varepsilon_{11}^e = 0 \tag{12}$$

where γ_1(D_1,D_2) = {1+H(D_2)[H(D_1)-1]}η(D_1)+{1+H(D_1)[H(D_2)-1]}η(D_2) and γ_2(D_1,D_2) = D_1+D_2.

The Heaveside functions employed above are given by:

$$H(D_i) = 1 \quad \text{if} \quad D_i > 0; \qquad H(D_i) = 0 \quad \text{if} \quad D_i = 0 \quad (i = 1, 2) \tag{13}$$

The η(D_1) and η(D_2) functions are defined, respectively, for the tension and compression cases, assuming for the first one that there was no previous damage of compression affecting the present traction damage variable D_1 and analogously, for the second one that has not had previous damage of tension affecting variable D_2. Accordingly, the functions η(D_1) and η(D_2) can be written as:

$$\eta(D_1) = \frac{-D_1 + \sqrt{3 - 2D_1^2}}{3}; \qquad \eta(D_2) = \frac{-D_2 + \sqrt{3 - 2D_2^2}}{3} \tag{14}$$

3.2 Criterion and evolution laws of damage

As it has already been pointed out, in the model formulation the damage induces anisotropy in the concrete. Therefore, is convenient to separate the damage criteria into two criteria: the first one is used only to indicate damage beginning, or that the material is no longer isotropic and the second one is used for loading and unloading when the material is already considered as transverse isotropic. This second criterion identifies if there is or not evolution of the damage variables. For identifying the damage beginning it is suggested a criterion that compares the complementary elastic strain energy W_e^*, which is computed locally considering the medium as initially virgin, isotropic and purely elastic, with a certain reference value Y_{0T}, or Y_{0C}, obtained from experimental tests of uniaxial tension, or compression, respectively. Accordingly, the criterion for initial activation of damage processes in tension or compression is given by:

$$f_{T,C}(\sigma) = W_e^* - Y_{0T,0C} < 0 \tag{15}$$

then $D_T = 0$ (i. e., $D_1 = D_4 = 0$) for dominant tension states or $D_C = 0$ (i. e., $D_2 = D_3 = D_5 = 0$) for dominant compression states, where the response regime of the material is linear elastic and isotropic.

The reference values Y_{0T} and Y_{0C} are model parameters defined by the following expressions:

$$Y_{0T} = \frac{\sigma_{0T}^2}{2E_0} \quad ; \quad Y_{0C} = \frac{\sigma_{0C}^2}{2E_0} \tag{16}$$

where σ_{0T} and σ_{0C} are the limit elastic stresses determined in uniaxial tension and compression regimes.

It is important to notice that the damaged medium presents a transverse isotropy plane in correspondence to the current damage level. Then, the complementary elastic energy of the damaged medium is expressed in different forms, depending on whether tension or compression strain states prevail. In the case of dominant tension states ($\mathrm{g}(\varepsilon,\mathrm{D}_T,\mathrm{D}_C) > 0$) the expression is the following one:

$$\begin{aligned} W_{e_+}^* = & \frac{\sigma_{11}^2}{2E_0(1-D_1)^2} + \frac{(\sigma_{22}^2+\sigma_{33}^2)}{2E_0} - \frac{\nu_0\sigma_{11}(\sigma_{22}+\sigma_{33})}{E_0(1-D_1)} - \frac{\nu_0\sigma_{22}\sigma_{33}}{E_0} + \\ & + \frac{(1+\nu_0)(\sigma_{12}^2+\sigma_{13}^2)}{E_0(1-D_4)^2(1-D_5)^2} + \frac{(1+\nu_0)\sigma_{23}^2}{E_0} \end{aligned} \tag{17}$$

On the other hand, for dominant compression states ($\mathrm{g}(\varepsilon,\mathrm{D}_T,\mathrm{D}_C) < 0$), the complementary elastic energy for a material with transverse isotropy induced by damage is expressed for:

$$\begin{aligned} W_{e_-}^* = & \frac{\sigma_{11}^2}{2E_0(1-D_2)^2} + \frac{(\sigma_{22}^2+\sigma_{33}^2)}{2E_0(1-D_3)^2} - \frac{\nu_0\sigma_{11}(\sigma_{22}+\sigma_{33})}{E_0(1-D_2)(1-D_3)} - \frac{\nu_0\sigma_{22}\sigma_{33}}{E_0(1-D_3)^2} + \\ & + \frac{(1+\nu_0)(\sigma_{12}^2+\sigma_{13}^2)}{E_0(1-D_4)^2(1-D_5)^2} + \frac{(1+\nu_0)\sigma_{23}^2}{E_0} \end{aligned} \tag{18}$$

The previous relations were written by assuming that the transverse isotropy plane is known and that a system of local coordinates has been adopted whose the direction 1 is perpendicular that plane.

If we consider then, such a general situation of damaged medium the loading and unloading criterion for predominant tension regime is represented by the following relationship:

$$f_T(\sigma) = W_{e_+}^* - Y_{0T}^* \leq 0 \tag{19}$$

where the reference value Y^*_{0T} is defined by the maximum complementary elastic energy computed throughout the damage process up to the current state, i. e.:

$$Y^*_{0T} = \text{MAX}(Y^*_{0T}, W^*_{e_+}) \tag{20}$$

For the damaged medium in predominant compression regime, the relationships are similar to the tension case.

In the case of loading, i. e., when$\dot{D}_T \neq 0$ or$\dot{D}_C \neq 0$, it is necessary to update the values of the scalar damage variables that appear in the D_T and D_C tensors, considering their evolution laws.

In a general way, the relationships that define the associated variables may be represented by:

$$Y_{T,C} = F(\sigma, \mathbf{E}_0, \mathbf{D}_{T,C}) \tag{21}$$

Taking also into account an implicit representation, the damage evolution laws may be given by:

$$\dot{D}_{T,C} = F^*(Y_{T,C}, b_{T,C}) \tag{22}$$

where $b_{T,C}$ are groups of parameters incorporated in the evolution laws of D_T or D_C.Observe that in case of monotonic loading, the Eq. (22) can be integrated directly. However, the set of relationships formed by $Y_{T,C}$ and $D_{T,C}$ leads to an implicit system whose solution can be obtained by an iterative procedure.

Considering just the case of monotonic loading, the evolution laws proposed for the scalar damage variables are resulting of fittings on experimental results and present similar characteristics the those one described in both works: [7] and [1]. The general form proposed is:

$$D_i = 1 - \frac{1 + A_i}{A_i + \exp\left[B_i(Y_i - Y_{0i})\right]} \quad \text{with} \quad i = 1,5 \tag{23}$$

where A_i, B_i and Y_{0i} are parameters that must be identified. The parameters Y_{0i} are understood as initial limits for the damage activation, the same ones used in Eq. (15).

3.3 Criterion for the definition of the transverse isotropy local plane of the material

Initially it is established a general criterion for the existence of the transverse isotropy plane. In this work is proposed that the transverse isotropy due to damage only arises if positive strain rates exist at least in one of the principal directions. After assuming such proposition as valid, some rules to identify its location must be defined. First of all, considering a strain state in which one of the strain rates is no-null or has sign contrary to the others, the following rule is applied:

"In the principal strain space, if two of the three strain rates are extension, shortening or null, the plane defined by them will be the transverse isotropy local plane of the material."

The uniaxial tension is an example of this case where the transverse isotropy plane is perpendicular to the tension stress direction.

However, there are some cases that won't follow this rule. For example, the plane strain state in which the no-null strains have contrary signs. In this case, the first rule is not able to identify the transverse isotropy local plane of the material, so that a second rule must be applied:

"In a plane strain state, where the principal strain rates in the plane have contrary signs, the transverse isotropy local plane of the material is defined by both direction of the principal strain which is permanently null and the direction of the strain whose rate is positive."

Another particular case occurs when all principal strain rates are positive. For those states it is valid a third rule, which assumes that the direction of larger extension is perpendicular to the transverse isotropy local plane of the material.

3.4 Damage model with permanent strains – uniaxial version

Experimental observations indicate that the permanent strains are not negligible in the unloading situations. Some damage models take into account in their formulations those strains associating them exclusively to the damage phenomenon. In this context, it is important mentioning the models proposed for [2, 10, 11], among others.

Taking into account just the uniaxial cases, the formulation of the proposed model is then extended to incorporate permanent strains, which are assumed to appear after the damage has been activated.

Assuming, for simplicity, that the permanent strains are composed exclusively by volumetric strains, as it has already been considered in others works [12], and taking into account the unilateral effect, the evolution law for the permanent strains results:

$$\dot{\varepsilon}^p = \left(\frac{\beta_T}{(1 - D_1)^2} \dot{D}_1 + \frac{\beta_C}{(1 - D_2)^2} \dot{D}_2 \right) \mathbf{I} \tag{24}$$

Observe that β_T and β_C are parameters directly related to the evolutions of permanent strains induced by damage in tension and in compression, respectively.

4 Numerical results

In the first example it is considered an uniaxial tension test of a concrete specimen ($E_0 = 15600$ MPa e $\nu_0 = 0.2$). The experimental results were presented by [8] in a test denominated PIED ("Pour Identifier L'Endommagement Diffus"). The model parameters were obtained by calibration of the normal stress x strain in the direction of the applied load curve, resulting in: $A_1 = 70$, $B_1 = 22110$ MPa^{-1} and $Y_{01} = 1.5 \times 10^{-5}$ MPa.

In Figure (1) the numerical response obtained from the proposed model is compared with the experimental response of the PIED test. Observe that in the case of transverse strains (directions 2 and 3) it has been shown just the results referred to the identified parameters, not having experimental data for comparison in this case.

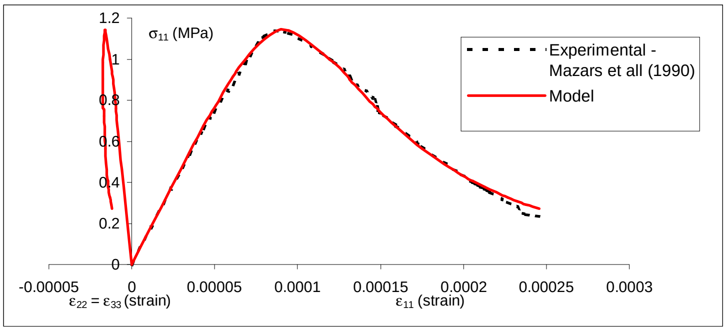


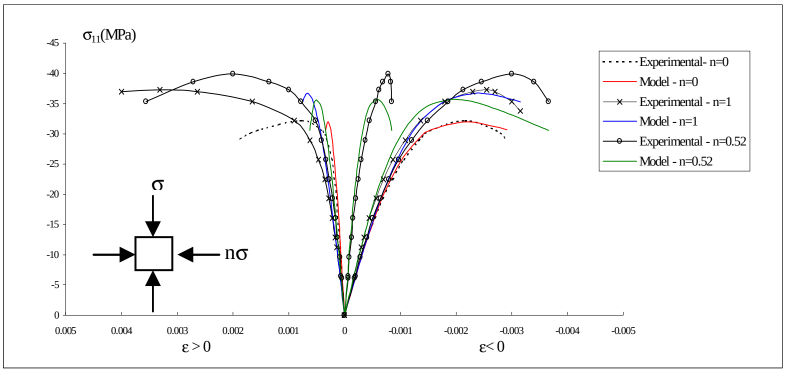
Figure 1: Simulation of the uniaxial tension test: experimental and numerical results.

It can be noted in Figure (1) that the group of parameters identified for the model allows reproducing quite well the experimental data. Regarding the transverse strains reproduced by the model, it is important noting that they are in correspondence with the form described in [14].

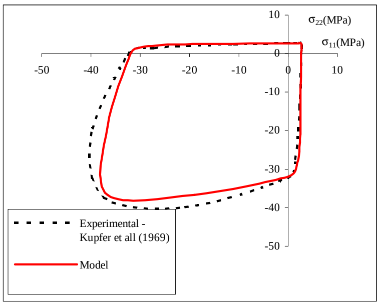
The model was also used in the simulation of biaxial and uniaxial tests, as carried out in Reference [5] for concrete specimens. The elasticity modulus and the Poisson's rate used in the numerical simulation were: $E_o = 31850$ MPa, $\nu_o = 0.2$. It has to be pointed out that in cases of more complex stress states, the calibration of the group of parameters, which appear in the evolution laws of the damage variables involved, require also uniaxial and biaxial tests. In particular, the parameters related to the variables D_1 and D_2 were obtained from uniaxial tension and compression tests, respectively: $A_1 = 69.4$, $B_1 = 9500$ MPa^{-1}, $Y_{01} = 0.8 \text{ x } 10^{-4}$ MPa, $A_2 = -0.80$, $B_2 = 0.90$ MPa^{-1} and $Y_{02} = 0.2 \text{ x } 10^{-2}$ MPa.

However the parameters related to D_3 were obtained from the biaxial compression test ($\sigma_{11} = \sigma_{22}$) by calibrating the stress-strain experimental curves in the directions 1 and 2. The resulting parameters were: $A_3 = -0.60$, $B_3 = 1.305$ MPa^{-1} and $Y_{03} = Y_{02} = 0.2 \text{ x } 10^{-2}$ MPa.

The Figure (2a) shows the comparison between the stress $x$strain experimental curves and the ones obtained with the proposed model for uniaxial and biaxial compression tests, considering different loading levels. It is worth pointing out that other load combinations, for example, biaxial compression $\sigma_{22} = 0.52\ \sigma_{11}$, compression-tension, were simulated with the parameters A_i , B_i and Y_{0i} described above. The experimental and numerical results are quite similar in the cases of the direct compression strains. However, the results of transverse strains underestimate the experimental ductility. This is due to the fact that at the region close to the pick stress, the residual strains start to play an important part in the concrete behavior. Another interesting result is the failure domain (Fig. (2b)), characterized by the stress picks obtained from the model and referred to uniaxial and biaxial loadings. It can be observed that the model is able to predict the failure domain satisfactorily.



(a)



(b)

Figure 2: Simulation of uniaxial and biaxial tests: a) experimental and numerical results; b) failure domain.

Let us consider now the application of the model in the numerical simulation of a test carried out as in Reference [9] for concrete specimens (E_0 = 23250 MPa and $\nu_0 = 0.2$) subjected to a triaxial compression loading according to the relationship $\sigma_{11} \quad < 0$; $\sigma_{22} = 0.10\ \sigma_{11}$; $\sigma_{33} = 0.05\sigma_{11}$ (Fig. (3). As the experimental response in uniaxial and biaxial compression of this concrete is not known, one has decided to do the calibration of the model parameters by using the stress x strain experimental curve in the direction 1, being the resulting parameters related to the damage variables D_2 and D_3 equal to: A_2 = -0.80, B_2 = 0.15 MPa^{-1}, A_3 = -0.60, B_3 = 1.305 MPa^{-1} and $Y_{02} = Y_{03}$ = 0.2 x 10^{-2} MPa.

It is observed, in Figure (3), a good accuracy of the numerical results related to the curve $\sigma_{11} \quad x \quad \varepsilon_{11}$ according to the experimental response. However, the results referred to the transverse strains, once again are not able to reproduce the evident experimental ductility. Neverthe-

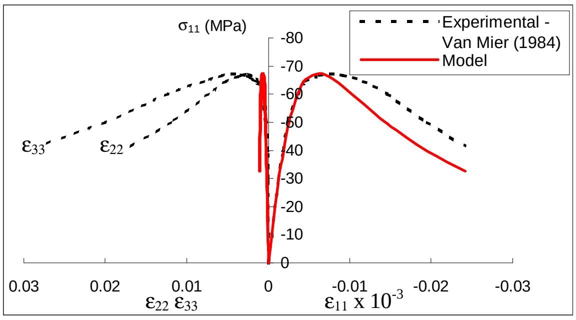


Figure 3: Simulation of triaxial compression test: experimental and numerical results.

less, the model is capable of simulating, in a quite reasonable way, the experimental behaviour up to the ultimate stress.

The uniaxial version of the model considering permanent strains was used in the simulation of the uniaxial compression test given by [5]. The parameters were obtained by the calibrating the stress $x$strain experimental curve resulting in: A_2 = 0,70, B_2 = 5,50 MPa^{-1}, Y_{02} = 2,0 x 10^{-3} MPa and β_C = 1,58 x 10^{-3}. The Figure (4) evidences that the incorporation of the permanent strains improves the capture of the transverse strains. Besides, the proposed model correctly predicts the change in sign of the volumetric strain, which is experimentally observed.

5 Application on the reinforced concrete frame analysis

The one-dimensional version of the model proposed here was implemented in a program for bars structures analysis with finite layered elements (EFICoS – "Eléments Finis à Couches Superposées"), which already contains the damage models of [7] and [1].

In the layered elements it is assumed as hypotheses that the distortions strains are negligible. The assumed to govern the concrete layers behavior are the ones reported above and for the longitudinal reinforcement bars, an elastoplastic behavior is admitted.

In the transversal section, a certain layer can contain steel and concrete. By assuming a perfect adherence between the materials, it is defined, for each layer, an elastic modulus and a inelastic strain equivalents, by using homogenization rule.

The frame geometry in Reference [13] and its reinforcement distribution are illustrated in Fig. 5. The concrete used in the frame has elasticity modulus E_c = 30400 MPa; the steel has E_a = 192500 MPa, yielding stress of 418 MPa and ultimate stress of 596 MPa. A bilinear elastoplastic model was adopted with a reduced elasticity modulus in the second branch: E_{a2} = 0,009 E_a. Table 1 contains the parameters values.

In the experimental test, it was initially applied an axial load of 700 kN for each column, which was maintained constant during all the lateral load application. This force was applied in increments up to the frame ultimate load.

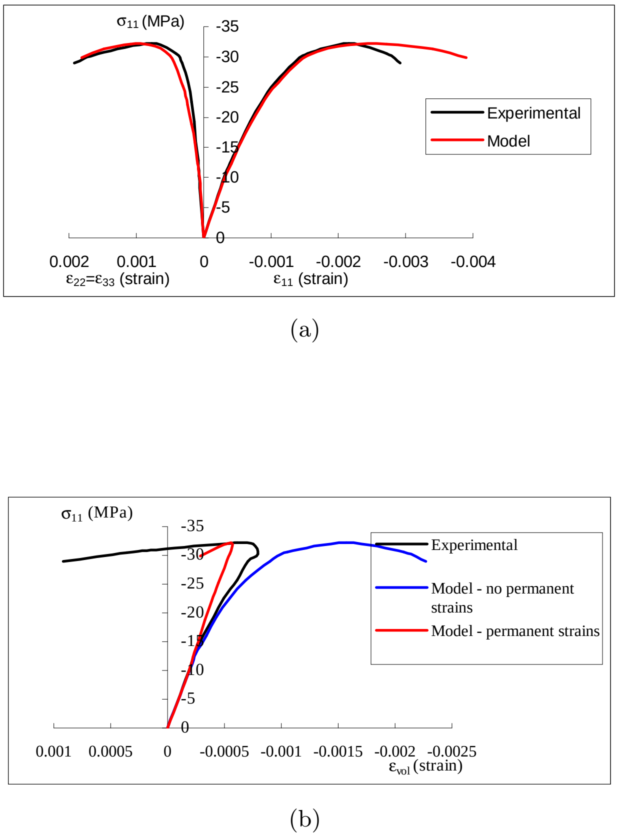


Figure 4: Simulation of uniaxial compression test: experimental and numerical results

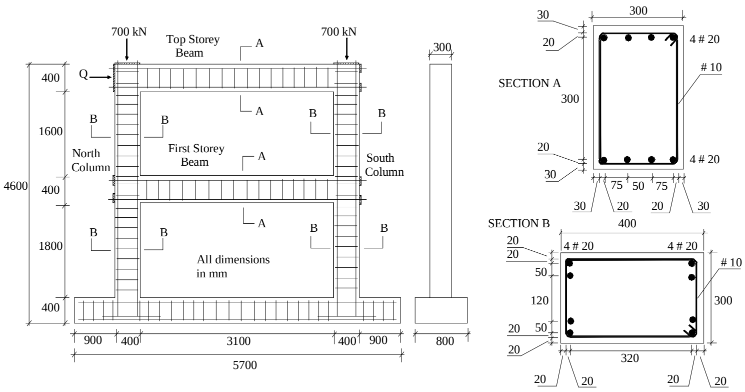


Figure 5: Geometry and reinforced details of the frame

Table 1: Parameters values of the models: Mazars [8], La Borderie [9] and proposed model.

Mazars [8]	La Borderie [9]		Model	
	Tension	Compression	Tension	Compression
$A_T = 0.995$	$Y_{01} = 3.05\text{x}10^{-4}$ MPa	$Y_{02} = 0.5\text{x}10^{-2}$ MPa	$Y_{01} = 0.72\text{x}10^{-4}$ MPa	$Y_{02} = 0.17\text{x}10^{-2}$ MPa
$B_T = 8000$	$A_1 = 3.50\text{x}10^{+3}$ MPa^{-1}	$A_2 = 6.80$ MPa^{-1}	$A_1 = 49$	$A_2 = 0.30$
$A_C = 0.85$	$B_1 = 0.95$	$B_2 = 0.7705$	$B_1 = 6560$ MPa^{-1}	$B_2 = 5.13$ MPa^{-1}
$B_C = 1050$	$\beta_1 = 1.00$ MPa	$\beta_2 = -10.00$ MPa	$\beta_T = 1 \text{ x } 10^{-6}$ MPa	$\beta_C = 1\text{x}10^{-3}$ MPa
ε_{d0}=0.00007	$\sigma_f = 3.50$ MPa			

In the numerical analysis, displacements increments were enforced in the application point of the horizontal force. The frame was discretized into 30 finite elements, 10 of which were used in the discretization of each column and 5 in each beam. The transversal sections were divided into 10 layers. The numerical and experimental responses are displayed in Figure (6), where the graphs represent the applied horizontal force x horizontal displacement relationship computed at the superior floor of the frame.

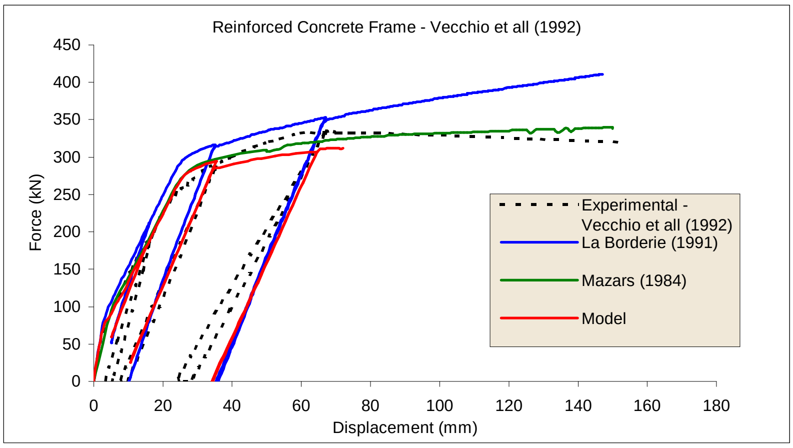


Figure 6: One-dimensional frame analysis: experimental and numerical results

The results obtained by the models have shown to be satisfactory in spite of the limited parametric identification. By considering permanent strains, both the La Borderie [9] and the proposed model were able to reproduce the residual displacements observed experimentally.

Note that both the Mazars' model [8] and the one proposed here can compute accurate responses, without requiring a very fine discretization.

The proposed model combined together with the discretizacion technique adopted to simulate the behavior of linear reinforced concrete structures has been shown to be efficient and accurate.

6 Conclusions

In this work, the formulation of constitutive models proposed in Reference [3], for anisotropic elastic materials that behave differently in tension and compression, has been extended in order to incorporate anisotropy and bimodular behavior induced by damage.

Among the aspects related to the CDM that were incorporated to the model derived from the mentioned formulation extension, it has to be pointed out the energy equivalence which leads to symmetric constitutive tensors. The bimodular character induced by damage was taken into account by the definition of two damage tensors, one for dominant tension states and another one for dominant compression states. Because of this bimodular character, it was necessary to define a criterion for dividing the strain space.

Energy criteria were introduced for identification of the damage beginning and its evolution processes. Another important aspect of the model is related to the determination of the transverse isotropy local plane of the material, which is necessary to establish the A tensor form used when the damage process is activated.

In a general way, the results represented here were satisfactory, mainly concerning the recovery of the pick stress values and of the concrete behavior when subjected to high stress values. The stress-strain curves referred to uniaxial tension or compression as well as the failure domain obtained for biaxial stress states, have shown the accuracy of the model when applied to simulate the mechanical behavior of the concrete. Besides, the results suggest that the residual strains can have an important influence in the pos-pick responses. To confirm that statement, numerical responses were presented considering permanent strains in the model formulation proposed originally.

The proposed model has shown quite efficient when dealing with bars structures. It is believed that some advantages of its employment related to the isotropic models, such as the selective stiffness deterioration combined together with the model capacity of simulating the concrete unilateral behavior, should be evident in two-dimensional and three-dimensional analyses, what encourage us to proceed with this formulation to deal with more complex structures in future works.

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Appendix

A Mathematical relationships

There are three fourth-order isotropic tensors linearly independents:

$$\mathbf{I} \otimes \mathbf{I} = \delta_{ij}\delta_{kl}(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.1}$$

$$\mathbf{I}\underline{\otimes}\mathbf{I} = \delta_{ik}\delta_{jl}(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.2}$$

$$\mathbf{I}\overline{\otimes}\mathbf{I} = \delta_{il}\delta_{jk}(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.3}$$

A tensor derived from the previous ones is the following:

$$\mathbf{I}\overline{\underline{\otimes}}\mathbf{I} = \frac{1}{2}\left[I\underline{\otimes}I + I\overline{\otimes}I\right] \tag{A.4}$$

In what follows, the characteristic properties of those fourth-order isotropic tensors are presented. In this context, let us consider a second-order tensor S, which is transformed by the fourth-order isotropic tensors:

$$(I \otimes I)\, S = (trS)\mathbf{I} = (\mathbf{S}.\mathbf{I})\mathbf{I} \tag{A.5}$$

$$(I\underline{\otimes}I)\, S = \mathbf{S} \tag{A.6}$$

$$(I\overline{\otimes}I)\, S = \mathbf{S}^T \tag{A.7}$$

$$(I\overline{\underline{\otimes}}I)\, S = \frac{1}{2}\left(S + S^T\right) \tag{A.8}$$

Taking into account the tensors properties, the following fourth-order tensors involved in the formulation can be derived:

$$\mathbf{A} \otimes \mathbf{A} = A_{ij}A_{kl}(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.9}$$

$$\mathbf{A} \otimes \mathbf{I} = A_{ij}\delta_{kl}(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.10}$$

$$\mathbf{I} \otimes \mathbf{A} = \delta_{ij}A_{kl}(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.11}$$

$$\mathbf{A}\overline{\underline{\otimes}}\mathbf{I} = \frac{1}{2}(A_{ik}\delta_{jl} + \delta_{il}A_{jk})(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.12}$$

$$\mathbf{I}\overline{\underline{\otimes}}\mathbf{A} = \frac{1}{2}(\delta_{ik}A_{jl} + A_{il}\delta_{jk})(e_i \otimes e_j \otimes e_k \otimes e_l) \tag{A.13}$$

It can also be summarized the operations for dyadic products applications including any second-order tensor, which were described in Curnier et al. [3]:

$$[R \otimes T]X = (X.T)R \qquad \forall \qquad \mathbf{X} \in \xi(\text{second order symmetric tensors space}) \tag{A.14}$$

$$[R\underline{\otimes}T]X = RXT^T; \tag{A.15}$$

$$[R\overline{\otimes}T]X = RX^TT^T \tag{A.16}$$

$$[R\overline{\underline{\otimes}}T]X = (RXT^T + TX^TR^T)/2 \qquad i.e. \qquad R\overline{\underline{\otimes}}T = [R\underline{\otimes}T + T\overline{\otimes}R]/2 \tag{A.17}$$