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The Unified Solution for a Beam of Rectangular Cross-Section with Different Higher-Order Shear Deformation Models

Abstract

The unified solution is studied for a beam of rectangular cross section. With the rotation defined in the average sense over the cross section, the kinematics with higher-order shear deformation models in axial displacement is first expressed in a unified form by using the fundamental higher-order term with some properties. The shear correction factor is then derived and discussed for the four commonly used higher-order shear deformation models including the third-order model, the sine model, the hyperbolic sine model and the exponential model. The unified solution is finally obtained for a beam subjected to an arbitrarily distributed load. The relation with that from the conventional beam theory is established, and therefore the difference is reasonably explained. A very good agreement with the elasticity theory validates the present solution.

Keywords

Fundamental higher-order term, Unified higher-order shear deformation model, Shear correction factor, Unified solution, Conventional beam theory.

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1 INTRODUCTION

As the basic structure, beams have been extensively used in fields such as architecture, mechanics, chemistry, aerospace and ocean engineering. Two approaches are often adopted in studying beams. One is to use the elasticity theory, two dimensional or three dimensional, while the other is to use a beam theory as a one-dimensional problem. In the first approach, the solution to a problem is derived strictly according to the elasticity theory. However, as we know, this approach can only offer analytic solutions to problems with simple geometries or loadings.

For the second approach, the situation is somewhat confused. To date, there have developed quite a number of beam theories. The Euler-Bernoulli theory (EBT) (Dym and Shames, 1973) is the first and classic beam theory in which the cross section is assumed normal to the neutral axis before and after deformation and hence the shear deformation of cross section is not taken into account. In contrast, the Timoshenko beam theory (TBT) (Timoshenko, 1922, 1921) permits a uniform shear deformation of cross section with a shear correction factor.

To improve the accuracy, higher-order shear deformation (HSD) models were proposed. For example, Levinson (1981) suggested a third-order rectangular beam model by taking the in-plane warping of cross section into account. Based on the same idea, Murthy (1981) used another definition of rotation, leading to the shear correction factor 5/6 rather than 2/3 by Levinson (1981). Following the kinematics of Levinson (1981), Bickford (1982) proposed a variationally consistent beam theory and Reddy (1984a, 1984b) further developed a higher-order differential governing equation for plates. The comprehensive numerical investigations on the accuracy of various HSD models were conducted by Rohwer (1992) which showed that the Murthy's (1981) and Reddy's (1984a, 1984b) theories are the best choices. Inspired by Murthy (1981) and Bickford (1982), Shi (2011, 2007) proposed an improved third-order shear deformation theory with a variationally consistent sixth-order governing equation and respective boundary conditions.

Rohwer (1992) indicated that the kinematics and the rotation play important roles in the HSD model. For the kinematics, besides the third-order shear deformation model (Levinson, 1981; Murthy, 1981), the trigonometric model (Akgöz and Civalek, 2014a, 2014b; Karama et al., 1998; Mantari et al., 2012; Touratier, 1991), the hyperbolic sine model (Akgöz and Civalek, 2015; Akavci and Tanrikulu, 2008; El Meiche et al., 2011; Soldatos, 1992) and the exponential model (Aydogdu, 2009; Karama et al., 2003) have been developed. For the rotation, two definitions are usually adopted in the existing work. As the first one, the rotation variable ψ is defined as the rotation at the neutral surface (e.g. Akgöz and Civalek, 2015; Akgöz and Civalek, 2014a, 2014b; Aydogdu, 2009; Bickford, 1982; El Meiche et al., 2011; Groh, 2015; Karama et al., 2003; Karama et al., 1998; Levison, 1981; Mantari et al., 2012; Qu, et al., 2013; Reddy, 1984a, 1984b; Simsek, 2010; Soldatos, 1992; Touratier, 1991; Viola, et al., 2013) while, as the second one, the rotation variable ϕ is defined as the average rotation over the cross section in some sense (e.g. Cowper, 1966; Murthy, 1981; Reissner, 1975; Shi, 2011, 2007). In this context, the beam theory in terms of ψ is called the conventional one.

Compared with the TBT, HSD models can not only reflect the warping of cross section, but give the shear correction factor in a straightforward manner. However, the TBT is amazing in evaluating deflection due to its sophisticated physics in the shear correction factor. In virtue of this, much attention was paid to evaluating the shear correction factor (e.g. Cowper, 1966; Dong et al., 2013, 2010; Gruttmann and Wagner, 2001; Gruttmann et al., 1999; Hutchinson, 2001; Jensen, 1983; Kaneko, 1975; Pai and Schulz, 1999).

In recent years, HSD models have been widely used in composite structures (Aydogdu, 2009; Karama et al., 2003; Mantari et al., 2012; Viola, et al., 2013) and functionally graded (FG) structures (El Meiche et al., 2011; Simsek, 2010). Much attention was also paid to real world applications by using beam theories. For example, Wang et al. (2008) studied the bending problems of micro- and nano-beams based on the Eringen nonlocal elasticity theory and the TBT, and found that the shear deformation and small-scale effect are significant in these problems. In addition, HSD models have been adopted to study small-scale structures (Akgöz and Civalek, 2015, 2014a, 2014b; Challamel, 2013, 2011; Wang et al., 2014) in nano- or micro-electro mechanical systems (NEMS or MEMS).

Though there have been massive researches on beam problems, some confusions are still pending. For example, what are the proper quantities in characterizing governing equations and boundary conditions of a beam problem? Are there any connections among the existing beam theories, HSD models, or shear correction factors? These are just the issues to touch on in the present work. To this end, the paper is outlined as follows. In Section 2, with the rotation defined in the average sense over the cross section, the beam theory is summarized, including derivation of the governing equations and the boundary conditions. In Section 3, the HSD model is expressed in a unified form for the kinematics in axial displacement by introducing the fundamental higher-order term with some properties. Four commonly used HSD models, viz. the third-order model, the sine model, the hyperbolic sine model and the exponential model, are then studied in detail, and the shear correction factors are finally calculated. In Section 4, the unified solution is derived and discussed. The concluding remarks are made in Section 5.

2 FUNDAMENTALS OF THE BEAM THEORY

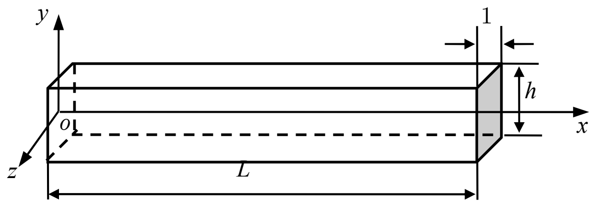


Figure 1: A straight beam of rectangular cross-section.

For a straight beam of rectangular cross-section, the coordinate system is shown in Figure 1. In the present work, according to the quantities in the plane elasticity theory, moment M, shear force Q, deflection w and rotation ϕ for a beam are respectively defined as

$$\begin{cases} M(x) = \int_A y\sigma_x dA \\ Q(x) = \int_A \tau_{xy} dA \\ w(x) = \dfrac{1}{A}\int_A v dA \\ \phi(x) = \dfrac{1}{I}\int_A yu dA \end{cases} \tag{1}$$

where σ_x , σ_y and τ_{xy} are the stress components in the x-y plane. $u(x, y)$ and $v(x, y)$ are respectively the axial and transverse displacements in this plane. A and I are respectively the area and the moment of inertia of cross section. It is easily seen that definitions for M, Q and w are the same as often while the definition for rotation is somewhat different.

In this context, in parallel to Eqs. (1), the beam theory will be also established from the plane elasticity theory. Ignoring body forces, the governing equations in the plane elasticity theory are

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{in the } x - \text{direction} \\ \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yx}}{\partial x} = 0 \quad \text{in the } y - \text{direction} \end{cases} \tag{2}$$

Taking the end of x=0 as an example, the corresponding boundary conditions are

$$\begin{cases} u(0,y) \text{ given or } \sigma_x(0,y) \text{ given} \quad \text{in the } x - \text{direction} \\ v(0,y) \text{ given or } \tau_{xy}(0,y) \text{ given} \quad \text{in the } y - \text{direction} \end{cases} \tag{3}$$

In the x-direction, integration of the first of Eqs. (2) and (3) weighted by y over the cross section yields

$$\frac{d}{dx}\int_A y\sigma_x dA + \int_A y \frac{\partial \tau_{xy}}{\partial y} dA = 0 \tag{4}$$

and

$$\int_A yudA = \int_A yu_0 dA \text{ or } \int_A y\sigma_x dA = \int_A y\sigma_0 dA \tag{5}$$

In a similar manner, in the y-direction, integrating the second of Eqs. (2) and (3) over the cross section, we have

$$\frac{d}{dx}\int_A \tau_{yx} dA + \int_A \frac{\partial \sigma_y}{\partial y} dA = 0 \tag{6}$$

and

$$\int_A vdA = \int_A v_0 dA \text{ or } \int_A \tau_{xy} dA = \int_A \tau_0 dA \tag{7}$$

Considering the first two of Eqs. (1), Eqs. (4) and (6) yield

$$\begin{cases} M' - Q = 0 \\ Q' + q(x) = 0 \end{cases} \tag{8}$$

with

$$q(x) = \int_A \frac{\partial \sigma_y}{\partial y} dA \tag{9}$$

In this context, the prime denotes the differentiation with respect to x.

Considering the definitions in Eqs. (1), Eqs. (5) and (7) yield

$$\begin{cases} \phi(0) \text{ given or } M(0) \text{ given} & \text{in the } x-\text{direction} \\ w(0) \text{ given or } Q(0) \text{ given} & \text{in the } y-\text{direction} \end{cases} \tag{10}$$

With Eqs. (10), all the practical boundary conditions in the *x*-direction and/or in the *y*-direction can be prescribed. For instance, we have

$$\begin{cases} \text{Free condition}: M(0) \text{ given and } Q(0) \text{ given} \\ \text{Simply supported condition}: M(0) \text{ given and } w(0)=0 \\ \text{Fixed condition}: \phi(0)=0 \text{ and } w(0)=0 \end{cases} \tag{11}$$

It should be noted that, due to the definitions in Eqs. (1), Eqs. (11) can also serve as the boundary conditions in the plane elasticity theory.

For a beam structure, the fundamental assumption is that $\sigma_y << \tau_{xy} << \sigma_x$ (e.g. see Ghugal and Sharma, 2011), so, from the first of Eqs. (1), in terms of ϕ, the moment can be further expressed as

$$M = EI\phi' \tag{12}$$

Together with Eqs. (8), the governing equations read

$$\begin{cases} EI\phi''' + q(x) = 0 \\ EI\phi'' = Q \end{cases} \tag{13}$$

3 THE KINEMATICS

3.1 The Unified Higher-Order Shear Deformation Model

In the second of Eqs. (13), Q must be expressed in terms of w(x) and/or ϕ(x). To this end, first, as often, the transverse displacement is assumed to be independent of the thickness coordinate y, and hence we have

$$v(x,y) \equiv w(x) \tag{14}$$

Next, the axial displacement is assumed to be expressed by

$$u(x,y) = y \cdot f_1(x) + g(y) \cdot f_R(x) \tag{15}$$

where y and g(y) are respectively called the first-order term and the fundamental higher-order term with respect to y while $f_1(x)$ and $f_R(x)$ are the corresponding coefficient functions with respect to x. It is worth noticing that g(y) is a pure higher-order term in this context.

Substituting Eq. (15) into the fourth of Eqs. (1) yields

$$\phi(x) = f_1(x) + \lambda \cdot f_R(x) \tag{16}$$

where

$$\lambda = (1/I)\int_A y g(y) dA \tag{17}$$

If we take $f_R(x) \equiv 0$, together with Eq. (16), Eq. (15) reduces to the first-order shear deformation model as

$$u(x,y) = y \cdot \phi(x) \tag{18}$$

For a higher-order model, upon eliminating $f_R(x)$ via Eq. (16), Eq. (15) becomes

$$u = y \cdot f_1(x) + [g(y)/\lambda](\phi(x) - f_1(x)) \tag{19}$$

Thus, considering Eq. (14), the shear strain over the cross section is

$$\gamma_{xy} = u_{,y} + v_{,x} = (f_1 + w') + [dg(y)/dy](\phi - f_1)/\lambda \tag{20}$$

For a beam of rectangular cross section, the shear stress free condition is often adopted on the top and bottom surfaces, which requires

$$\gamma_{xy}(x, \pm h/2) = 0 \tag{21}$$

So, from Eq. (20), we have

$$f_1 = \frac{c\phi + \lambda w'}{c - \lambda} \tag{22}$$

where

$$c = (dg/dy)\big|_{y=h/2} = (dg/dy)\big|_{y=-h/2} \tag{23}$$

Eventually, the axial displacement in Eq. (15) becomes

$$u = y \cdot \frac{c\phi + \lambda w'}{c - \lambda} - \frac{g(y)}{c - \lambda}(\phi + w') \tag{24}$$

Eq. (24) is termed as the unified HSD model for a beam corresponding to the fundamental higher-order term $g(y)$.

Thus far, from Eqs. (23) and considering the higher-order property, $g(y)$ has the following properties

$$\begin{cases} g(y)\big|_{y=0} = 0 \\ (dg/dy)\big|_{y=0} = 0 \\ (dg/dy)\big|_{y=h/2} = (dg/dy)\big|_{y=-h/2} \end{cases} \tag{25}$$

In terms of $w(x)$ and $\phi(x)$, the shear force $Q(x)$ is expressed as

$$Q(x) = \int_A G\gamma_{xy} dA = K_P GA\left(\phi + w'\right) \tag{26}$$

where K_P, the shear correction factor originally defined by Timoshenko (1922,1921), takes the form of

$$K_P = \frac{1}{A\left(c-\lambda\right)} \cdot \int_A \left[c - dg/dy\right] dA \tag{27}$$

It can be seen from Eq. (27) that the shear correction factor will be certainly determined as long as g(y) is known, rather than other procedures (e.g. Cowper, 1966; Hutchinson, 2001).

If we assume

$$\phi + w' \equiv 0 \tag{28}$$

Eq. (26) yields

$$Q(x) \equiv 0 \tag{29}$$

which implies that the shear effect cannot be not taken into account.

Accordingly, Eq. (24) reduces to

$$u\left(x, y\right) = -y \cdot w'(x) \tag{30}$$

which is the axial displacement assumption in the EBT (Dym and Shames,1973). Compared with the first-order shear deformation model in Eq. (18), the EBT in Eq. (30) is obtained under the additional assumption of Eq. (28), which is a more reasonable explanation than that of an infinite shear rigidity (e.g. Challamel, 2013).

3.2 Four Commonly Used HSD Models

One direct choice of the fundamental higher-order term is to take $g^A(y)=y^3$. This model is just the commonly used third-order model and denoted as Model-A in this context. Eq. (27) yields

$$K_P^A = 5/6 \tag{31}$$

In this paper, other three commonly used HSD models are studied as well.

3.2.1 The Sine Model – Model-B

Inspired by Touratier (1991) and considering Eqs. (25), as the sine model, the fundamental higher-order term is taken to be

$$g^B(y) = y - \frac{h}{\pi}\sin\left(\frac{\pi y}{h}\right) \tag{32}$$

Accordingly, from Eqs. (17) and (23), we have

$$\begin{cases} \lambda = 1 - \dfrac{24}{\pi^3} \\ c = 1 \end{cases} \tag{33}$$

So, the axial displacement in Eq. (24) yields

$$u^B = -yw' + \frac{\pi^2 h}{24} \sin\left(\frac{\pi y}{h}\right)(\phi + w') \tag{34}$$

Further, Eq. (27) yields

$$K_P^B = \pi^2 / 12 \tag{35}$$

3.2.2 The Hyperbolic Sine Model – Model-C

Inspired by El Meiche et al. (2011) and Soldatos (1992) and considering Eq. (25), as the hyperbolic sine model, the fundamental higher-order term is taken to be

$$g^C(y) = y - h \sinh\left(\frac{y}{h}\right) \tag{36}$$

Accordingly, from Eqs. (17) and (23), we have

$$\begin{cases} \lambda = 1 - 12 \cdot \left[\cosh(1/2) - 2 \cdot \sinh(1/2)\right] \\ c = 1 - \cosh(1/2) \end{cases} \tag{37}$$

So, the axial displacement in Eq. (24) yields

$$u^C = -yw' + \frac{y \cdot \cosh(1/2) - h \cdot \sinh(y/h)}{24 \cdot \sinh(1/2) - 11 \cdot \cosh(1/2)}(\phi + w') \tag{38}$$

Further, Eq. (27) yields

$$K_P^C = \frac{\cosh(1/2) - 2\sinh(1/2)}{24\sinh(1/2) - 11\cosh(1/2)} \tag{39}$$

3.2.3 The Exponential Model– Model-D

Inspired by Karama et al. (2003) and considering Eq. (25), as the exponential model, the fundamental higher-order term is taken to be

$$g^D(y) = y - y \cdot \exp\left[-2(y/h)^2\right] \tag{40}$$

Accordingly, from Eqs. (17) and (23), we have

$$\begin{cases} \lambda = 1 + 3 \cdot \exp(-1/2) - (3\sqrt{2\pi}/2) erf(\sqrt{2}/2) \\ c = 1 \end{cases} \tag{41}$$

where the Gauss error function is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2) ds \tag{42}$$

So, the axial displacement in Eq. (24) yields

$$u^D = -yw' + \frac{y \cdot \exp\left[-2 \cdot (y/h)^2\right](\phi + w')}{-3\exp(-1/2) + (3\sqrt{2\pi}/2) erf(\sqrt{2}/2)} \tag{43}$$

Further, Eq. (27) yields

$$K_P^D = \frac{\exp(-1/2)}{-3\exp(-1/2) + (3\sqrt{2\pi}/2) erf(\sqrt{2}/2)} \tag{44}$$

3.3 Comparison of the Four HSD Models

3.3.1 The First-Order Term and the Third-Order Term Through Taylor's Expansion

It is interesting to study the difference of the four HSD models by comparing the first two leading terms through Taylor's expansion.

For the third-order model, the axial displacement is

$$u^A(x, y) = y\phi + y\frac{1}{4}(w' + \phi) - \frac{y^3}{h^2}\frac{5}{3}(w' + \phi) \tag{45}$$

Thus, if the axial displacement is expanded in form of power series

$$u(x, y) = y\phi + yc_1(w' + \phi) - \frac{y^3}{h^2}c_3(w' + \phi) + O(y^5/h^5)(w' + \phi) \tag{46}$$

from Taylor's expansion, we have

$$\begin{cases} c_1^A = 1/4;\ c_3^A = 5/3 & \text{For Model - A} \\ c_1^B = \pi^3/24 - 1;\ c_3^B = \pi^5/144 & \text{For Model - B} \\ \begin{cases} c_1^C = \left[\cosh(1/2) - 1\right]/\left[24\sinh(1/2) - 11\cosh(1/2)\right] - 1 \\ c_3^C = 1/\left\{6\left[24\sinh(1/2) - 11\cosh(1/2)\right]\right\} \end{cases} & \text{For Model - C} \\ \begin{cases} c_1^D = 1/\left[-3\exp(-1/2) + \left(3\sqrt{2\pi}/2\right) erf\left(\sqrt{2}/2\right)\right] - 1 \\ c_3^D = 2/\left[-3\exp(-1/2) + \left(3\sqrt{2\pi}/2\right) erf\left(\sqrt{2}/2\right)\right] \end{cases} & \text{For Model - D} \end{cases} \tag{47}$$

The coefficients are summarized in Table 1. It is seen that c_3 increases with c_1.

Model type	c_1	c_3
Model-A	0.250	1.667
Model-B	0.292	2.125
Model-C	0.246	1.628
Model-D	0.338	2.676

Table 1: Comparison of the four HSD models in c_1 and c_3

3.3.2 Shear Correction Factors

As already indicated in Section 3.1, given g(y), the shear correction factor can be evaluated for the beam of rectangular cross section. Values of the shear correction factor for the four HSD models in Section 3.2 are summarized in Table 2. It is seen that they vary a bit with different HSD models. In addition, K_P becomes smaller if the HSD model (i.e. Model-D with the biggest c_3) is farther deviated from the first-order shear deformation model.

Model type	Model-A	Model-B	Model-C	Model-D
K_P	0.8333	0.8225	0.8343	0.8116
c_3	1.667	2.125	1.628	2.676

Table 2: Comparison of the four HSD models in K_P

From Eqs. (31) and (35), we can see that the commonly used shear correction factors can be reasonably explained by Eq. (27) in the manner of HSD models. For example, the TBT with $K_P = 5/6$ (e.g. Kaneko, 1975; Timoshenko, 1922, 1921; Murthy, 1981; Reissner, 1975; Shi, 2007, 2011) is in essential equivalent to the third-order model (i.e. Model-A) because of Eq. (31) while the Mindlin plate theory (Mindlin, 1951) with $K_P = \pi^2/12$ is in essential equivalent to the sine model (i.e. Model-B) because of Eq. (35).

4 THE UNIFIED SOLUTION

4.1 Derivation of the Unified Solution

In this section, the unified solution will be derived for the unified HSD model in Section 3. To this end, the governing equations are re-arranged as follows

$$\begin{cases} EI\phi''' + q(x) = 0 \\ EI\phi'' = K_P GA(\phi + w') \end{cases} \tag{48}$$

It is interesting to discuss the procedure of solving Eqs. (48). A straightforward procedure is to turn Eqs. (48) into the form

$$\begin{cases} \phi''' = -q(x)/EI \\ w' = \dfrac{EI}{K_P GA}\phi'' - \phi \end{cases} \tag{49}$$

From the first of Eqs. (49) - a third-order differential equation, the rotation ϕ can be first obtained, and then, from the second of Eqs. (49) – a first-order differential equation, the deflection w can be further obtained.

In this context, a more general approach is used instead. To this end, Eqs. (48) are re-expressed as

$$\begin{cases} w^{(4)} = \dfrac{q(x)}{EI} - \dfrac{q''(x)}{K_P GA} \\ \phi = -w' - \dfrac{EI}{K_P GA}\left[w''' + \dfrac{q'(x)}{K_P GA} \right] \end{cases} \tag{50}$$

From the first of Eqs. (50), w is firstly obtained by solving a fourth-order differential equation, and ϕ is then directly obtained from the second of Eqs. (50). Eventually, we have

$$\begin{cases} w = \dfrac{1}{EI}\int_0^x dx \int_0^x dx \int_0^x dx \int_0^x q(x)dx - \dfrac{1}{K_P GA}\int_0^x dx \int_0^x q(x)dx + a_1 x^3 + a_2 x^2 + a_3 x + a_4 \\ \phi = -\dfrac{1}{EI}\int_0^x dx \int_0^x dx \int_0^x q(x)dx - 3a_1 x^2 - 2a_2 x - \left(a_3 + \dfrac{6EIa_1}{K_P GA} \right) \end{cases} \tag{51}$$

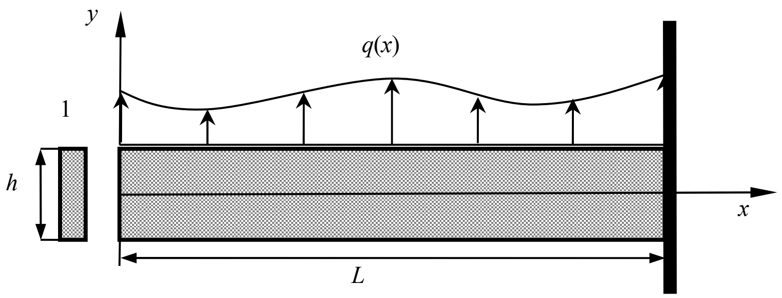


Figure 2: A cantilever beam under an arbitrarily distributed load

For the problem shown in Figure 2, the boundary conditions are

$$\begin{cases} \text{free condition at } x = 0: \quad EI\phi'(0) = 0 \;,\; K_P GA\left(\phi(0) + w'(0)\right) = 0 \\ \text{fixed condition at } x = L: \quad \phi(L) = 0 \;,\; w(L) = 0 \end{cases} \tag{52}$$

So, the integration constants in Eqs. (51) are finally determined as

$$\begin{cases} a_1 = -\dfrac{1}{6EI}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ a_2 = -\dfrac{1}{2EI}\left[\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ a_3 = -\dfrac{1}{EI}\left[\int_0^x dx\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=L} + \dfrac{L}{EI}\left[\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ \quad +\dfrac{L^2}{2EI}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} + \dfrac{1}{KGA}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ a_4 = -\dfrac{1}{EI}\left[\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=L} + \dfrac{L}{EI}\left[\int_0^x dx\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=L} \\ \quad -\dfrac{L^2}{2EI}\left[\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=0} - \dfrac{L^3}{3EI}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ \quad +\dfrac{1}{K_P GA}\left[\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=L} - \dfrac{L}{K_P GA}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \end{cases} \tag{53}$$

Eqs. (51) and (53) are the unified solution for the beam problem shown in Figure 2.

The unified solution is also derived in Appendix A (see Eqs. (A.24) and (A.26)) by using the conventional beam theory. It is seen that Eqs. (A.24) will turn into Eqs. (51) if $\alpha_T=1$ (and hence $K_T=K_P$ from Eq. (A.13)). However, the fact is that α_T is actually much less than unity for all the HSD models (see Table A.1), leading to major difference between K_T and K_P. This may be the reason why K_T was not recognized as the shear correction factor in the previous HSD models (e.g. Akgöz and Civalek, 2014a, 2014b; El Meiche et al., 2011; Huang et al., 2013; Levinson, 1987, 1981; Simsek, 2010; Touratier, 1991) despite of the definition already made as early as in 1920s (Timoshenko, 1922,1921).

4.2 Comparison of the Results

With the unified solution, the results can be comparatively studied in detail. For this purpose, the special case of q=const is considered. From Eqs. (51) and (53), it is not difficult to obtain the unified solution to this case as

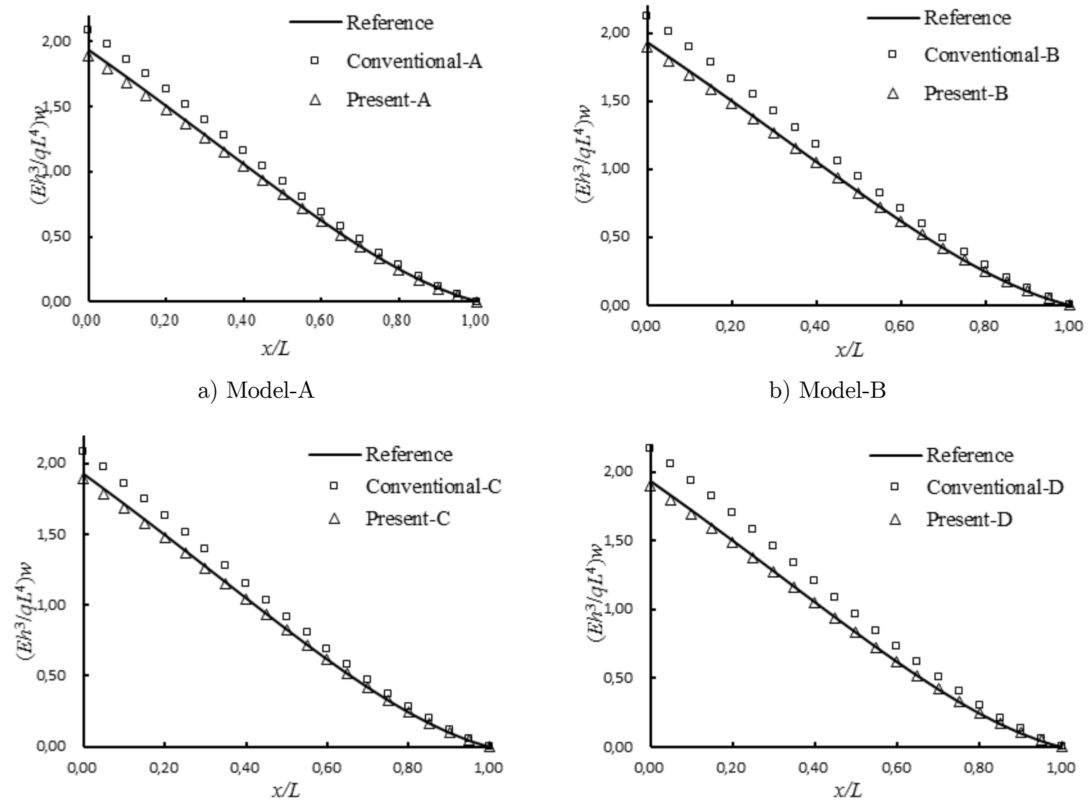
$$\begin{cases} w = \dfrac{q}{24EI}\left(x^4 - 4L^3x + 3L^4\right) - \dfrac{q}{2K_P GA}\left(x^2 - L^2\right) \\ \phi = -\dfrac{q}{6EI}\left(x^3 - L^3\right) \end{cases} \tag{54}$$

On the other hand, applying the boundary conditions in Eqs. (11) to the problem shown in Figure 2 with q=const, the elasticity solution to this plane stress problem is obtained as

$$\begin{cases} w_{Ref}^{q} = \dfrac{1}{A}\int v da = qL^4\left[\left(x/L\right)^4 - 4x/L + 3\right]\Big/\left(24EI\right) - qL^2h^2\left[7\left(x/L\right)^2 - 12x/L + 5\right]\Big/\left(120EI\right) \\ \qquad - qL^2\left[5\left(x/L\right)^2 + 12\left(x/L\right) - 17\right]\Big/\left(20GA\right) \\ u_{Ref}^{q} = y\left[qL^3\left[1-\left(x/L\right)^3\right]\Big/\left(6EI\right) + qLh^2\left(3x/L - 4\right)\big/\left(40EI\right) + 3qL\left(4 - 5x/L\right)\big/\left(20GA\right)\right] \\ \qquad - qLh^3\left(x/L - 1\right)\big/\left(24EI\right) + qLh\left(x/L - 1\right)\big/\left(4GA\right) + qxy^3\big/\left(6EI\right) + qxy^3\big/\left(GAh^2\right) \end{cases} \tag{55}$$

The deflections from different solutions are plotted in Figure 3 for the four HSD models in which “Reference”, “Conventional” and “Present” denote the solution from Eqs. (55), (A.27) and (54), respectively. It is seen that the current solution can always agree better with the reference for all the four HSD models than the conventional solution.

With Eqs. (51), the axial displacement (i.e. the warping of the cross section) can also be obtained through Eq. (24). The variation with y at x=L is plotted in Figure 4. It is seen that the warping from all the four HSD models are in considerably good agreement with the reference for the present solution. However, due to the apparent difference between K_P and K_T (see Table 2 and Table A.1), the conventional solution greatly deviates.



a) Model-A b) Model-B

c) Model-C d) Model-D

Figure 3: Comparison of deflections from the four HSD models.

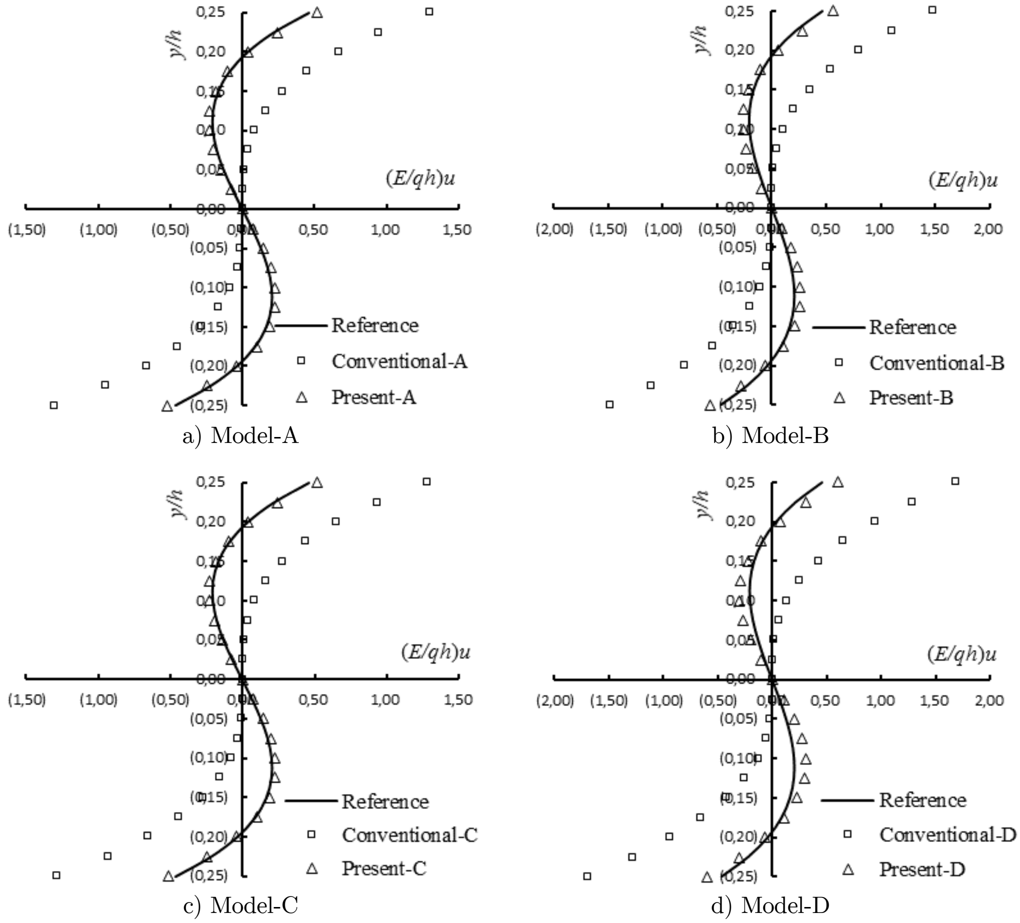


Figure 4: Comparison of the warping of cross section from the four HSD models.

5 CONCLUSIONS AND FUTURE WORK

In this paper, with the definitions of average deflection and average rotation, the governing equations and the boundary conditions were derived from the plane elasticity theory. Based on the kinematics in axial displacement, the unified HSD model was proposed by using the fundamental higher-order term with some properties. The shear correction factor was then derived. The unified solution was finally obtained for the unified HSD model subjected to an arbitrarily distributed load, and compared with the conventional one. From the current work, following conclusions can be made.

1) The definition of rotation plays an important role for the HSD models.
2) The kinematics of HSD models in axial displacement can be expressed by using the fundamental higher-order term with some properties.
3) Given a loading, the unified solution can be derived for a beam with different fundamental higher-order terms in axial displacement.
4) Given the fundamental higher-order term, the shear correction factors can be reasonably obtained.

The beam theory used in the current work is a lower-order (fourth-order) one in terms of deflection w. Future work will focus on a variationally consistent higher-order (sixth-order) beam theory and the shear correction factor of arbitrary cross section. Application to small scale problems and extension to a plate problem are also prospective.

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APPENDIX A: THE CONVENTIONAL BEAM THEORY WITH THE UNIFIED HSD MODEL AND THE UNIFIED SOLUTION

A.1 Governing Equations and Boundary Conditions

In the conventional beam theory, deflection $\tilde{w}(x)$ and rotation $\psi(x)$ are defined as (e.g. Timoshenko and Goodier, 2004)

$$\begin{cases} \tilde{w}(x) = v(x,0) \\ \psi(x) = (\partial u/\partial y)\big|_{y=0} \end{cases} \tag{A.1}$$

With the assumption of Eq. (15), $\tilde{w}(x)$ is identical to $w(x)$.

Together with the properties in the first two of Eqs. (25) for g(y), Eq. (15) is still mathematically valid. Thus, the shear strain is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left[f_1(x) + \tilde{w}'(x)\right] + \left(dg/dy\right)\cdot f_{\mathrm{R}}(x) \tag{A.2}$$

Considering the shear stress free condition on the top and bottom surfaces

$$\gamma_{xy}(x, \pm h/2) = \left[f_1(x) + \tilde{w}'(x)\right] + c\cdot f_{\mathrm{R}}(x) = 0 \tag{A.3}$$

we have

$$f_{\mathrm{R}}(x) = -\frac{1}{c}\left[f_1(x) + \tilde{w}'(x)\right] \tag{A.4}$$

where c is as defined in Eq. (23) and hence the property in the third of Eqs. (25) stand also for g(y).

From the definition in the second of Eqs. (A.1) and considering the third of Eqs. (25), we have

$$\psi(x) = f_1(x) \tag{A.5}$$

With Eqs. (A.4) and (A.5), in terms of $\tilde{w}(x)$ and $\psi(x)$, the axial displacement is eventually expressed as

$$\begin{aligned} u(x,y) &= y \cdot \psi - \left(g(y)/c\right)\left(\psi + \tilde{w}'\right) \\ &= -y \cdot \tilde{w}' + \left[y - \left(g(y)/c\right)\right]\left(\psi + \tilde{w}'\right) \end{aligned} \tag{A.6}$$

As often, Eq. (A.6) can also be re-expressed as

$$u(x,y) = -y \cdot \tilde{w}' + S(y)\left(\psi + \tilde{w}'\right) \tag{A.7}$$

with

$$S(y) = \left[y - \left(g(y)/c\right)\right] \tag{A.8}$$

Based on Eqs. (25), $S(y)$ has following properties

$$\begin{cases} S(y)\big|_{y=0} = 0 \\ \left(dS/dy\right)\big|_{y=0} = 1 \\ \left(dS/dy\right)\big|_{y=\pm\frac{h}{2}} = 0 \end{cases} \tag{A.9}$$

From Eq. (A.7), we further obtain

$$Q = K_T GA(\psi + \tilde{w}') \tag{A.10}$$

with

$$K_T = \frac{1}{A}\int_\Omega dS(y)/dy\, dA \tag{A.11}$$

In addition, in terms of $\tilde{w}(x)$ and $\psi(x)$ (Aydogdu , 2009; Akavci and Tanrikulu, 2008; El Meiche et al., 2011; Karama et al., 1998, 2003; Levinson, 1981; Mantari et al., 2012; Soldatos, 1992; Touratier, 1991), moment M can be expressed in a unified form as

$$M = -EI\tilde{w}'' + \alpha_T EI\left(\psi' + \tilde{w}''\right) \tag{A.12}$$

with

$$\alpha_T = \left(1/I\right) \cdot \int_\Omega y \cdot S(y) dA \tag{A.13}$$

Interestingly, we can obtain the following relationship

$$K_T / \alpha_T = K_P \tag{A.14}$$

From Eqs. (A.10) and (A.12), in terms of $\tilde{w}(x)$ and $\psi(x)$, the governing equations are

$$\begin{cases} K_T AG\left(\psi' + \tilde{w}''\right) + q\left(x\right) = 0 \\ -EI\tilde{w}''' + \alpha_T EI\left(\psi'' + \tilde{w}'''\right) = K_T AG\left(\psi + \tilde{w}'\right) \end{cases} \tag{A.15}$$

with the boundary conditions as [e.g. Levinson, 1981]

$$\begin{cases} \text{Free condition}: \ M = M_0 \ \text{ and } \ Q = Q_0 \\ \text{Simply supported condition}: \ M = M_0 \ \text{ and } \ \tilde{w} = \tilde{w}_0 = 0 \\ \text{Fixed condition}: \ \psi = \psi_0 = 0 \ \text{ and } \ \tilde{w} = \tilde{w}_0 = 0 \end{cases} \tag{A.16}$$

A.2The Four Commonly Used HSD Models

(1) The third-order model – Model-A (Levinson, 1981)

In the third order shear deformation model, we take $g^A(y)=y^3$, and hence $c=3h^2/4$ from Eq. (23). Then, Eq. (A.8) yields

$$S^A(y) = y\left(1 - \frac{4y^2}{3h^2}\right) \tag{A.17}$$

(2) The sine model – Model-B (Touratier, 1991)

In this HSD model, $g^B(y)$ takes the form of Eq. (32), and hence $c=1$ from Eq. (23). Then, Eq. (A.8) yields

$$S^B(y) = \frac{h}{\pi}\sin\left(\frac{\pi y}{h}\right) \tag{A.18}$$

(3) The hyperbolic sine model – Model-C (Soldatos, 1992)

In this HSD model, $g^C(y)$ takes the form of Eq. (36), and hence c=1-cosh(1/2) from Eq. (23). Then, Eq. (A.8) yields

$$S^C(y) = \frac{y\cosh\left(1/2\right) - h\sinh\left(y/h\right)}{\cosh\left(1/2\right) - 1} \tag{A.19}$$

It should be noted that Eq. (A.19) firstly derived according to Eqs. (A.8) corrects the original form in Soldatos (1992).

(4) The exponential model – Model-D (Karama et al., 2003)

In this HSD model, $g^D(y)$ takes the form of Eq. (40), and hence $c=1$ from Eq. (23). Then, Eq. (A.8) yields

$$S^D(y) = y\exp\left(-2y^2/h^2\right) \tag{A.20}$$

A.3 Shear Correction Factors

From Eq. (A.11), for the four HSD models, it is immediate to obtain

$$\begin{cases} K_T^A = 2/3 \\ K_T^B = 2/\pi \\ K_T^C = \dfrac{\cosh(1/2) - 2\sinh(1/2)}{\cosh\left(1/2\right) - 1} \\ K_T^D = \exp\left(-1/2\right) \end{cases} \tag{A.21}$$

In addition, from Eq. (A.13), we have

$$\begin{cases} {\alpha_T}^A = 4/5 \\ {\alpha_T}^B = 24/\pi^3 \\ {\alpha_T}^C = \dfrac{24\sinh(1/2) - 11\cosh(1/2)}{\cosh\left(1/2\right) - 1} \\ {\alpha_T}^D = -3\exp(-1/2) + \left(3\sqrt{2\pi}/2\right) erf\left(\sqrt{2}/2\right) \end{cases} \tag{A.22}$$

The values for K_T and α_T are summarized in Table A.1. Compared with K_P in Table 2, Eq. (A.14) can also be validated by the four HSD models.

Model type	Model-A	Model-B	Model-C	Model-D
K_T	0.6667	0.6366	0.6694	0.6065
α_T	0.8000	0.7740	0.8024	0.7473

Table A.1: Comparison of the four HSD models in K_T and α_T.

A.4 The Unified Solution

For sake of the unified solution, Eqs. (A. 15) are further re-expressed as

$$\begin{cases} \tilde{w}^{(4)} = -\dfrac{\alpha_T q''(x)}{K_T AG} + \dfrac{q(x)}{EI} \\ \psi = -\dfrac{EI}{K_T AG}\left[\alpha_T \dfrac{q'(x)}{K_T AG} + \tilde{w}'''\right] - \tilde{w}' \end{cases} \tag{A.23}$$

Considering Eq. (A.14), the first of Eq. (A.23) is identical to the first of Eq. (50). From Eq. (A.23), it is not difficult to obtain the unified solution as

$$\begin{cases} \tilde{w} = \dfrac{1}{EI}\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x q(x)dx - \dfrac{\alpha_T}{K_T AG}\int_0^x dx\int_0^x q(x)dx + b_1x^3 + b_2x^2 + b_3x + b_4 \\ \psi = -\dfrac{1}{EI}\int_0^x dx\int_0^x dx\int_0^x q(x)dx + \dfrac{\alpha_T - 1}{K_T AG}\int_0^x q(x)dx - 3b_1x^2 - 2b_2x - \left(\dfrac{6EI}{K_T AG}b_1 + b_3\right) \end{cases} \tag{A.24}$$

For the problem shown in Figure 2, from Eqs. (A.16), the corresponding boundary conditions for the conventional beam theory are

$$\begin{cases} \text{Free condition at } x=0: \quad -EI\tilde{w}''(0) + \alpha_T EI\left(\psi'(0) + \tilde{w}''(0)\right) = 0 \\ \qquad\qquad K_T GA\left(\psi(0) + \tilde{w}'(0)\right) = 0 \\ \text{Fixed condition at } x=L: \quad \psi(L) = 0 \ , \ \tilde{w}(L) = 0 \end{cases} \tag{A.25}$$

Thus, the integration constants in Eqs. (A.24) are determined as

$$\begin{cases} b_1 = -\dfrac{1}{6EI}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ b_2 = -\dfrac{1}{2EI}\left[\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ b_3 = -\dfrac{1}{EI}\left[\int_0^x dx\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=L} + \dfrac{L}{EI}\left[\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=0} + \dfrac{L^2}{2EI}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ \quad + \dfrac{\alpha_T - 1}{K_T AG}\left[\int_0^x q(x)dx\right]\Bigg|_{x=L} + \dfrac{1}{K_T AG}\left[\int_0^x q(x)dx\right]_{x=0} \\ b_4 = -\dfrac{1}{EI}\left[\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=L} + \dfrac{L}{EI}\left[\int_0^x dx\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=L} \\ \quad - \dfrac{L^2}{2EI}\left[\int_0^x dx\int_0^x q(x)dx\right]\Bigg|_{x=0} - \dfrac{L^3}{3EI}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \\ \quad + \dfrac{\alpha_T}{K_T AG}\left[\int_0^x dx\int_0^x q(x)dx\right]_{x=L} - \dfrac{\alpha_T - 1}{K_T AG}L\left[\int_0^x q(x)dx\right]\Bigg|_{x=L} - \dfrac{L}{K_T AG}\left[\int_0^x q(x)dx\right]\Bigg|_{x=0} \end{cases} \tag{A.26}$$

For the case of q=const, the unified solution to the unified HSD model is

$$\begin{cases} w_T = \dfrac{q}{24EI}\left(x^4 - 4L^3x + 3L^4\right) - \dfrac{\alpha_T q}{2K_T AG}\left(x^2 - L^2\right) + \dfrac{\left(\alpha_T - 1\right)qL}{K_T AG}\left(x - L\right) \\ \psi = -\dfrac{q}{6EI}\left(x^3 - L^3\right) + \dfrac{\left(\alpha_T - 1\right)q}{K_T AG}\left(x - L\right) \end{cases} \tag{A.27}$$

Int erestingly, Eqs. (A.27) will turn into Eqs. (54) if $\alpha_T=1$ (hence $K_T=K_P$ from Eq. (A.13)).