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Buckling and Vibration of Functionally Graded Non-uniform Circular Plates Resting on Winkler Foundation

Abstract

An investigation on the effect of uniform tensile in-plane force on the radially symmetric vibratory characteristics of functionally graded circular plates of linearly varying thickness along radial direction and resting on a Winkler foundation has been carried out on the basis of classical plate theory. The non-homogeneous mechanical properties of the plate are assumed to be graded through the thickness and described by a power function of the thickness coordinate. The governing differential equation for such a plate model has been obtained using Hamilton's principle. The differential transform method has been employed to obtain the frequency equations for simply supported and clamped boundary conditions. The effect of various parameters like volume fraction index, taper parameter, foundation parameter and the in-plane force parameter has been analysed on the first three natural frequencies of vibration. By allowing the frequency to approach zero, the critical buckling loads for both the plates have been computed. Three-dimensional mode shapes for specified plates have been plotted. Comparison with existing results has been made.

Keywords

Functionally graded circular plates, Buckling, Differential transform, Winkler foundation.

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1 INTRODUCTION

Now-a-days, technologists are able to tailor advanced materials by mixing two or more materials to get the desired mechanical properties along one/ more direction(s) due to their extensive demand in many fields of modern engineering applications. Functionally graded materials (FGMs) are the recent innovation of this class. In classic ceramic/metal FGMs, the ceramic phase offers thermal barrier effects and protects the metal from corrosion and oxidation and the FGM is toughened and strengthened by the metallic constituent. By nature, such materials are microscopically non-homogeneous, in which the material properties vary continuously in a certain manner either along a line or in a plane or in space. The plate type structural elements of FGMs have their wide applications in solar energy generators, nuclear energy reactors, space shuttle etc. and particularly in defence - as penetration resistant materials used for armour plates and bullet-proof vests. This necessitates to study their dynamic behaviour with a fair amount of accuracy.

Since the introduction of FGMs by Japanese scientists in 1984 (Koizumi, 1993), a number of researches dealing with the vibration characteristics of FGM plates of various geometries has been made due to their practical importance. A critical review of the work upto 2012 has been given by Jha et al. (2013). In addition to this, natural frequencies of functionally graded anisotropic rectangular plates have been studied by Batra and Jin (2005) using first-order shear deformation theory coupled with the finite element method. The first and third-order shear deformation plate theories have been used by Ferreira et al. (2006) in analyzing the free vibrations of rectangular FGM plates using global collocation method by approximating the trial solution with multiquadric radial basis functions. The same method has been employed by Roque et al. (2007) to present the free vibration analysis of FGM rectangular plates using a refined theory. Zhao et al. (2009) used element-free *kp*-Ritz method for free vibration analysis of rectangular and skew plates with different boundary conditions taking four types of functionally graded materials on the basis of first-order shear deformation theory. Liu et al. (2010) have analysed the free vibration of FGM rectangular plates with in-plane material inhomogeneity using Fourier series expansion and a particular integration technique on the basis of classical plate theory. Navier solution method has been used by Jha et al. (2012) to analyse the free vibration of FG rectangular plates employing higher order shear and normal deformation theory. The vibration behaviour of rectangular FG plates with non-ideal boundary conditions has been studied by Najafizadeh et al. (2012) using Levy method and Lindstedt-Poincare perturbation technique. The vibration solutions for FGM rectangular plate with in-plane material inhomogeneity have been obtained by Uymaz et al. (2012) using Ritz method and assuming the displacement functions in the form of Chebyshev polynomials on the basis of five-degree-of-freedom shear deformable plate theory. Thai et al. (2013) have developed an efficient shear deformation theory for vibration of rectangular FGM plates which accounts for a quadratic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the surfaces of the plate without using shear correction factors. Recently, Dozio (2014) has derived first-known exact solutions for free vibration of thick and moderately thick FGM rectangular plates with at least on pair of opposite edges simply supported on the basis of a family of two-dimensional shear and normal deformation theories with variable order. Very recently, the natural frequencies of FGM nanoplates are analyzed by Zare et al. (2015) for different combinations of boundary conditions by introducing a new exact solution method.

Under normal working conditions, plate type structures may be subjected to in-plane stressing arising from hydrostatic, centrifugal and thermal stresses (Brayan (1890-91), Leissa (1982), Wang et al. (2004)), which may induce buckling, a phenomenon which is highly undesirable. Numerous studies dealing with the effect of uniaxial/biaxial in-plane forces on the vibrational behaviour of FGM plates are available in the literature and reported by Feldman and Aboudi (1997), Najafizadeh and Heydari (2008), Mahdavian (2009), Baferani et al. (2012), Javaheri and Eslami (2002), Prakash and Ganapati (2006), Zhao (2009), Zhang et al. (2014). Among these, Feldman and

Aboudi (1997) analysed the elastic-bifurcational buckling of FG rectangular plates under in-plane compressive loading employing a combination of micromechanical and structural approach. The closed form solutions for the axisymmetric buckling of FG circular plates under uniform radial compression have been obtained by Najafizadeh and Heydari (2008) on the basis of higher order shear deformation plate theory. Baferani et al. (2012) have used Bessel functions to analyze the symmetric and asymmetric buckling modes of functionally graded annular plates under mechanical and thermal loads. Thermal buckling of FGM rectangular plates on the basis of classical plate theory for four different types of thermal loading has been presented by Javaheri and Eslami (2002) using Galerkin's method. Finite element method has been applied by Prakash and Ganapati (2006) to analyse the asymmetric thermal buckling and vibration characteristics of FGM circular plates. Recently, Zhang et al. (2014) have used local Kringing meshless method for the mechanical and thermal buckling analysis of rectangular FGM plates on the basis of first-order shear deformation plate theory.

Plates with tapered thickness are broadly used in many engineering structural elements such as turbine disks, aircraft wings and clutches etc. With appropriate variation of thickness, these plates can have significantly greater efficiency for vibration as compared to the plate of uniform thickness and also provide the advantage of reduction in weight and size. Several researches have been made to study the vibrational behaviour of functionally graded plates of varying thickness. Exact element method has been used by Efraim and Eigenberger (2007) for the vibration analysis of thick annular isotropic and FGM plates of three forms of thickness variation: linear, quadratically concave and quadratically convex. Naei et al. (2007) have presented the buckling analysis of radially loaded FGM circular plate of linearly varying thickness using finite-element method. Free vibration analysis of functionally graded thick annular plates with linear and quadratic thickness variation along the radial direction is investigated by Tajeddini and Ohadi (2011) using the polynomial-Ritz method. Recently, the free vibrations of FGM circular plates of linearly varying thickness under axisymmetric condition have been analysed by Shamekhi (2013) using a meshless method in which point interpolation approach is employed for constructing the shape functions for Galerkin weak form formulation.

The problem of plates resting on an elastic foundation has achieved great importance in modern technology and foundation engineering. Airport runways, submerged floating tunnels, bridge decks, building footings, reinforced-concrete pavements of highways, railway tracks, buried pipelines and foundation of storage tanks etc. are some direct applications of the foundations. From the viewpoint of practical utility, the commonly used elastic foundation is Winkler's model. In this model, the foundation is virtually replaced by mutually independent, closely spaced linear elastic springs providing the resistance and reaction at every point which is taken to be proportional to the deflection at that point. This consideration leads to satisfactory results in a variety of problems. In this regard, numerous studies analyzing the effect of Winkler foundation on the dynamic behaviour of non-FGM plates are available in the literature and the recent ones are reported by Gupta et al. (2006), Hsu (2010), Li and Yuan (2011), Kägo and Lellep (2013), Zhong et al. (2014), Ghaheri et al. (2014), to mention a few. However, regarding FGM plates, a very limited work is available and done by Amini et al. (2009), Kumar and Lal (2013), Kiani and Eslami (2013), Joodaky (2013), Fallah (2013), Chakraverty and Pradhan (2014). Of these, Amini et al. (2009) have provided the exact three-dimensional vibration results for rectangular FGM plates resting on Winkler foundation by employing Chebyshev polynomials and Ritz's method. Recently, Kumar and Lal (2013) predicted the natural frequencies for axisymmetric vibrations of two-directional FG annular plates resting on Winkler foundation using differential quadrature method and Chebyshev collocation technique.

In the present study, a semi analytical approach: differential transform method proposed by Zhou (1986), has been employed to study the effect of in-plane force on the axisymmetric vibrations of functionally graded circular plate of linearly varying thickness and resting on a Winkler foundation. According to this method, the governing differential equation of motion for the present model of the plate gets reduced to a recurrence relation. Use of this recurrence relation in the transformed boundary conditions together with the regularity condition, one obtains a set of two algebraic equations. These resulting equations have been solved using MATLAB to get the frequencies. The material properties i.e. Young's modulus and density are assumed to be graded in the thickness direction and these properties vary according to a power-law in terms of volume fractions of the constituents. The natural frequencies are obtained for clamped and simply supported boundary conditions with different values of volume fraction indexn g, in-plane force parameter N, taper parameter γ and foundation parameter K_f. Critical buckling loads for varying values of plate parameter have been computed. Three-dimensional mode shapes for the first three modes of vibration have been presented for the specified plates. A comparison of results has been given.

2 MATHEMATICAL FORMULATION

Consider a FGM circular plate of radius a, thickness $h(r)$, Young's modulus $E(z)$, density ρ and subjected to uniform in-plane tensile force N_0. Let the plate be referred to a cylindrical polar coordinate system (R, θ, z), $z = 0$ being the middle plane of the plate. The top and bottom surfaces are $z = +h/2$ and $z = -h/2$, respectively. The line $R = 0$ is the axis of the plate. The equation of motion governing the transverse axisymmetric vibration i.e. $\partial()/\partial\theta$ of the present model (Figure 1) is given by (Leissa, (1969)):

$$\begin{aligned} D\, w_{,RRRR} + \frac{2}{R}\left[D + R\, D_{,R}\right] w_{,RRR} + \frac{1}{R^2}\left[-D + R(2+\nu)\, D_{,R} + R^2(D_{,RR} - N_0)\right] w_{,RR} \\ + \frac{1}{R^3}\left[D - R\, D_{,R} + R^2(\nu\, D_{,RR} - N_0)\right] w_{,R} + k_f\, w + \rho\, h\, w_{,tt} = 0 \end{aligned} \qquad (1)$$

where w is the transverse deflection, D the flexural rigidity, ν the Poisson's ratio and a comma followed by a suffix denotes the partial derivative with respect to that variable.

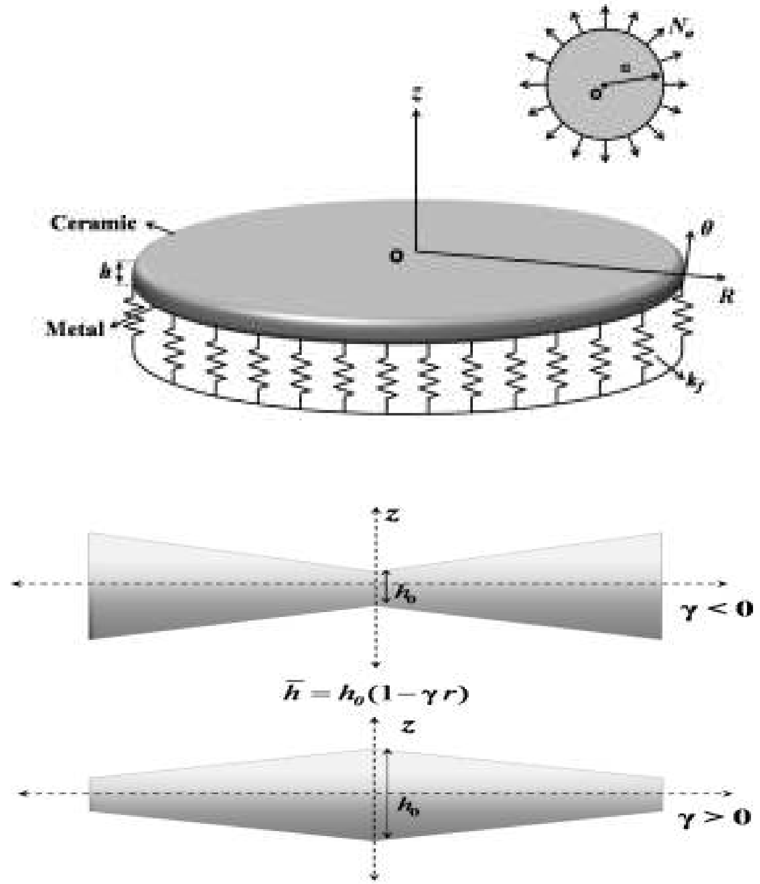


Figure 1: Geometry and cross-section of tapered FGM circular plate under uniform tensile load N_0 and resting on Winkler foundation.

For a harmonic solution, the deflection w can be expressed as

$$w(R,t) = W(R)e^{i\omega t} \tag{2}$$

where ω is the radian frequency and $i = \sqrt{-1}$. The Eq. (1) reduces to

$$\begin{aligned} D W_{,RRRR} &+ \frac{2}{R}\left[D + R\,D_{,R}\right]W_{,RRR} + \frac{1}{R^2}\left[-D + R(2+\nu)\,D_{,R} + R^2 D_{,RR}\right]W_{,RR} \\ &+ \frac{1}{R^3}\left[D - R\,D_{,R} + R^2\,\nu\,D_{,RR}\right]W_{,R} - N_0\,W_{,RR} \\ &- \frac{N_0}{R}W_{,R} + k_f\,W - \rho\,h\;\omega^2\,W = 0 \end{aligned} \tag{3}$$

Assuming that the top and bottom surfaces of the plate are ceramic and metal-rich, respectively, for which the variations of the Young's modulus $E(z)$ and the density $\rho(z)$ in the thickness direction are taken as follows (Dong, (2008)):

$$E(z) = (E_c - E_m)V_c(z) + E_m \tag{4}$$

$$\rho(z) = (\rho_c - \rho_m)V_c(z) + \rho_m \tag{5}$$

where E_c, ρ_c and E_m, ρ_m denote the Young's modulus and the density of ceramic and metal constituents, respectively and $V_c(z)$ is the volume fraction of ceramic as follows:

$$V_C(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^g \tag{6}$$

where g is used for the volume fraction index of the material.
The flexural rigidity and mass density of the plate material are given by

$$D = \frac{1}{1-\nu^2}\int_{-h/2}^{h/2} E(z)\, z^2 dz \tag{7}$$

$$\rho = \frac{1}{h}\int_{-h/2}^{h/2} \rho(z)\, dz \tag{8}$$

Using Eqs. (4-6) into Eq. (7) and Eq. (8), we obtain

$$\begin{aligned} D &= \frac{1}{1-\nu^2}\int_{-h/2}^{h/2}\left[(E_c - E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^g + E_m\right] z^2 dz \\ &= \frac{h^3}{1-\nu^2}\left[(E_c - E_m)\frac{g^2+g+2}{4(g+1)(g+2)(g+3)} + \frac{E_m}{12}\right] \end{aligned} \tag{9}$$

$$\rho = \frac{1}{h}\int_{-h/2}^{h/2}\left[(\rho_c - \rho_m)\left(\frac{z}{h}+\frac{1}{2}\right)^g + \rho_m\right] dz \;= \frac{\rho_c - \rho_m}{g+1} + \rho_m \;= \frac{\rho_c + \rho_m g}{g+1} \tag{10}$$

Introducing the non-dimensional variables $r = R/a$, $f = W/a$, $\bar{h} = h/a$, Eq. (3) now reduces to

$$\begin{aligned} &D f_{,rrrr} + \frac{2}{r}\left[D + r D_{,r}\right] f_{,rrr} + \frac{1}{r^2}\left[-D + r(2+\nu) D_{,r} + r^2 D_{,rr}\right] f_{,rr} \\ &+ \frac{1}{r^3}\left[D - r D_{,r} + r^2 \nu D_{,rr}\right] f_{,r} - N_0 a^2 f_{,rr} - \frac{N_0}{r} a^2 f_{,r} + k_f a^4 f = \rho a^4 \omega^2 f \bar{h} \end{aligned} \tag{11}$$

Assuming linear variation in the thickness i.e. $\bar{h} = h_0\left(1-\gamma r\right)$, γ being the taper parameter and h_0 is the non-dimensional thickness of the plate at the center. Substituting the values of D and ρ from Eq. (9) and Eq. (10) into Eq. (11), we get

$$\begin{aligned} & r^3\left(1-\gamma r\right)^3 B f_{,rrrr} + 2r^2\left(\left(1-\gamma r\right)^3 - 3\gamma r\left(1-\gamma r\right)^2\right) B f_{,rrr} \\ & \quad + rB\left(-\left(1-\gamma r\right)^3 - 3\gamma r\left(2+\nu\right)\left(1-\gamma r\right)^2 + 6r^2\alpha^2\left(1-\gamma r\right)\right) f_{,rr} \\ & \quad + B\left(\left(1-\gamma r\right)^3 + 3r\alpha\left(1-\gamma r\right)^2\right) f_{,r} \\ & \quad - N r^3 f_{,rr} - N r^2 f_{,r} + K_f r^3 f = r^3\Omega^2 A\left(1-\gamma r\right) f \end{aligned} \tag{12}$$

Where

$$D = D^* B(1-\gamma r)^3 a^3, \; N = \frac{N_0}{D^*}a^2, \; K_f = \frac{k_f}{D^*}a^4, \; \Omega^2 = \frac{\rho_c h_0 a^4}{D^*}\omega^2, \; D^* = \frac{E_c h_0^{\,3}}{12(1-\nu^2)},$$
$$A = \left(\frac{\rho_c + \rho_m g}{\rho_c(g+1)}\right), \quad B = \left[3\left(1-\frac{E_m}{E_c}\right)\frac{g^2+g+2}{(g+1)(g+2)(g+3)} + \frac{E_m}{E_c}\right]$$

Eq. (12) is a fourth-order differential equation with variable coefficients whose exact solution is not possible. The approximate solution with appropriate boundary and regularity conditions has been obtained employing differential transform method.

2.1 Boundary and regularity conditions

Following Zhou (1986), the relations which should be satisfied for simply supported/clamped plate at the boundary and regularity condition at the centre are given as follows:

(i) *Simply-supported edge*

$$f(1) = 0, \;\; M_r\,|_{r=1} = \left\{-D\left[\frac{d^2 f}{dr^2} + \nu\left(\frac{1}{r}\frac{df}{dr}\right)\right]\right\}_{r=1} = 0 \tag{13}$$

(ii) *Clamped edge*

$$f(1) = 0, \;\; \frac{df}{dr}\,|_{r=1} = 0 \tag{14}$$

(iii) *Regularity conditions at the centre ($r=0$) of the plate*

$$\frac{df}{dr}\,|_{r=0} = 0, \qquad Q_r\,|_{r=0} = \left[D\left(\frac{d^3 f}{dr^3} + \frac{1}{r}\frac{d^2 f}{dr^2} - \frac{1}{r^2}\frac{df}{dr}\right) + D_{,r}\left(\frac{d^2 f}{dr^2} + \frac{\nu}{r}\frac{df}{dr}\right)\right]_{r=0} = 0 \tag{15}$$

where M_r is the radial bending moment and Q_r the radial shear force.

3 METHOD OF SOLUTION

3.1 Description of the method

According to differential transform method Zhou (1986), an analytic function $f(\mathrm{r})$ in a domain $\left|r-r_0\right| \le a$ is expressed as a power series about the point r_0. The differential transform of its k^{th} derivative is given by

$$F_k = \frac{1}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} \tag{16}$$

The inverse transformation of the function F_k is defined as

$$f(r) = \sum_{k=0}^{\infty}(r-r_0)^k F_k \tag{17}$$

Combining the above two expressions, we have,

$$f(r) = \sum_{k=0}^{\infty}\frac{(r-r_0)^k}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} \tag{18}$$

In actual applications, the function $f(\mathrm{r})$ is expressed by a finite series. So, Eq. (18) may be written as:

$$f(r) = \sum_{k=0}^{n}\frac{(r-r_0)^k}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} \tag{19}$$

The convergence of the natural frequencies decides the value of n. Some basic theorems which are frequently used in the practical problems are given in Table 1.

Original Functions	Transformed Functions
$f(r) = g(r) \pm h(r)$	$F_k = G_k + H_k$
$f(r) = \lambda g(r)$	$F_k = \lambda G_k$
$f(r) = g(r)h(r)$	$F_k = \sum_{l=0}^{k} G_l H_{k-l}$
$f(r) = \frac{d^n g(r)}{dr^n}$	$F_k = \frac{(k+n)!}{k!} G_{k+n}$
$f(r) = r^n$	$F_k = \delta(k-n) = \begin{cases} 1 & k=n \\ 0 & k \ne n \end{cases}$

Table 1: Transformation Rules for one-dimensional DTM.

3.2 Transformation of the governing differential equation

Applying the transformation rules given in Table 1, the transformed form of the governing differential equation (12) around $r_0 = 0$ can be written as

$$\begin{aligned} F_{k+1} = \frac{1}{\left(k^2-1\right)^2} . [& 3\,\gamma\, k\left(k-1\right)\left(k^2-k-1+\nu\right)F_k \\ & + \{3\,\gamma\left(k-4\right)\left(k-3\right)\left(k-2\right)\left(k-1\right) - 6\,\nu\,\gamma^2\left(k-2\right)\left(k-1\right) \\ & - 3\gamma^2\left(k-1\right)\left(6\,k^2-25\,k+2\nu k-2\nu\right) + \frac{N}{B}(k-1)^2\}\, F_{k-1} \\ & + \gamma^3\left(k-2\right)\{k^3-4k^2+(2+3\nu)k-3\,\nu+1\}\, F_{k-2} \\ & + \frac{(\Omega^2 A - K_f)}{B} F_{k-3} - \gamma\,\Omega^2\,\frac{A}{B} F_{k-4}] \end{aligned} \tag{20}$$

3.3 Transformation of the boundary/regularity conditions

By applying transformations rules given in Table 1, the Eqs. (13, 14, 15) becomes:

Simply-supported edge condition:

$$\sum_{k=0}^{n} F_k = 0, \quad \sum_{k=0}^{n} [k(k-1)+\nu\, k] F_k = 0 \tag{21}$$

Clamped edge condition:

$$\sum_{k=0}^{n} F_k = 0, \quad \sum_{k=0}^{n} k\, F_k = 0 \tag{22}$$

Regularity condition:

$$F_1 = 0, \;\; F_3 = \frac{2}{3}\gamma(1+\nu)F_2 \tag{23}$$

4 FREQUENCY EQUATIONS

Since, the subscripts of the $F-$terms should be non-negative, so in Eq. (20), the subscript k should starts with 3. Starting with $k=3$ in Eq. (20), we get a recursive relation i.e. F_4 is determined in terms of F_0 and F_2, F_5 in terms of F_2 and F_4 and so on. Therefore, all the F terms can be expressed in terms of F_0 and F_2. Now, applying the boundary condition (21) on the resulted F_k expressions, we get the following equations:

$$\begin{aligned} \Phi_{11}^{(m)}(\Omega)F_0 + \Phi_{12}^{(m)}(\Omega)F_2 = 0 \\ \Phi_{21}^{(m)}(\Omega)F_0 + \Phi_{22}^{(m)}(\Omega)F_2 = 0 \end{aligned} \tag{24}$$

where $\Phi_{11}^{(m)}, \Phi_{12}^{(m)}, \Phi_{21}^{(m)}$ and $\Phi_{22}^{(m)}$ are polynomials in Ω of degree m where $m = 2n$. Eq. (24) can be expressed in matrix form as follows:

$$\begin{bmatrix} \Phi_{11}^{(m)}(\Omega) & \Phi_{12}^{(m)}(\Omega) \\ \Phi_{21}^{(m)}(\Omega) & \Phi_{22}^{(m)}(\Omega) \end{bmatrix} \begin{Bmatrix} F_0 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{25}$$

For a non-trivial solution of Eq. (25), the frequency determinant must vanish and hence

$$\begin{vmatrix} \Phi_{11}^{(m)}(\Omega) & \Phi_{12}^{(m)}(\Omega) \\ \Phi_{21}^{(m)}(\Omega) & \Phi_{22}^{(m)}(\Omega) \end{vmatrix} = 0 \tag{26}$$

Similarly, for clamped edge condition, the corresponding frequency determinant is given by

$$\begin{vmatrix} \Psi_{11}^{(m)}(\Omega) & \Psi_{12}^{(m)}(\Omega) \\ \Psi_{21}^{(m)}(\Omega) & \Psi_{22}^{(m)}(\Omega) \end{vmatrix} = 0 \tag{27}$$

where $\Psi_{11}^{(m)}$, $\Psi_{12}^{(m)}$, $\Psi_{21}^{(m)}$ and $\Psi_{22}^{(m)}$ are polynomials in Ω of degree m.

5 NUMERICAL RESULTS AND DISCUSSION

The frequency Eqs. (26) and (27) provide the values of the frequency parameter Ω . The lowest three roots of these equations have been obtained using MATLAB to investigate the influence of in-plane force parameter N , taper parameter γ , foundation parameter K_f and volume fraction index g on the frequency parameter Ω for both the boundary conditions. In the present analysis, the values of Young's modulus and density for aluminium as metal and alumina as ceramic constituents are taken from Dong (2008), as follows:

$$E_m = 70 \text{ GPa}, \quad \rho_m = 2,702 \text{ kg/m}^3; \; E_c = 380 \text{ GPa}, \quad \rho_c = 3,800 \text{ kg/m}^3$$

The variation in the values of Poisson's ratio is assumed to be negligible all over the plate and its value is taken as $\nu = 0.3$. From the literature, the values of various parameters are taken as:

Volume fraction index $g = 0, 1, 3, 5$;
In-plane force parameter $N =$ -30, -20, -10, 0, 10, 20, 30;
Taper parameter γ = -0.5, -0.3, -0.1, 0, 0.1, 0.3, 0.5; and
Foundation parameter $K_f =$ 0, 10, 50, 100.

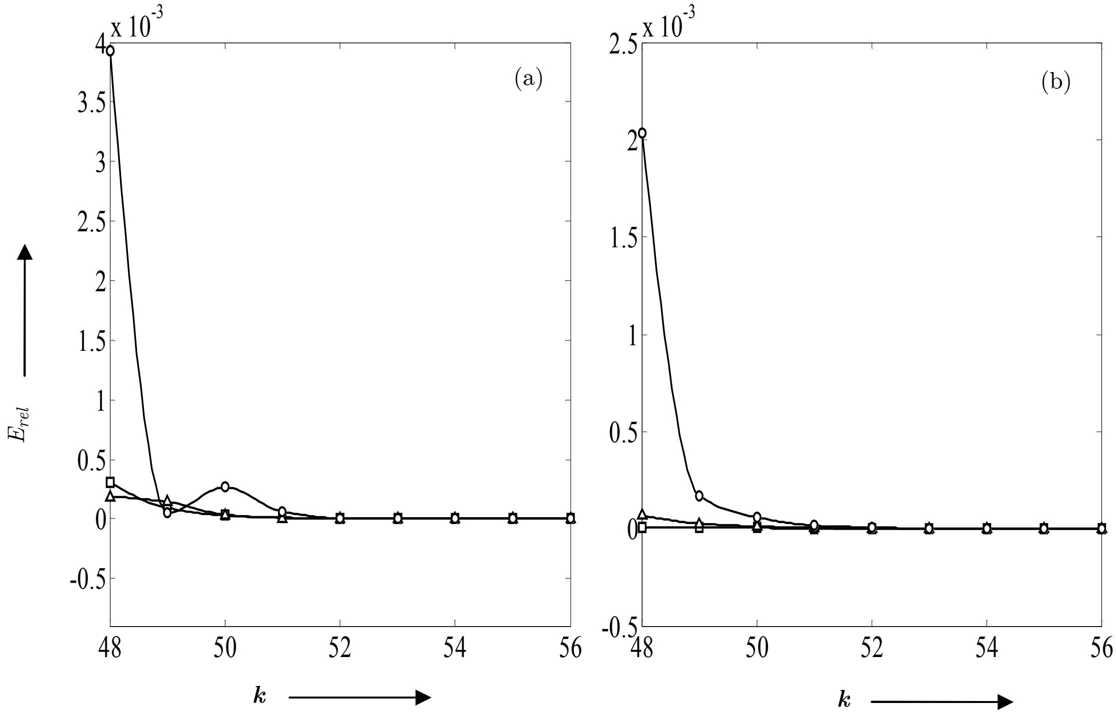


Figure 2: Relative error E_{rel} in Ω with no. of terms k: (a) simply supported plate (b) clamped plate for $g = 5$, $N = 30$, $\gamma = 0.5$, $K_f = 100$.□ , first mode; Δ, second mode; ○, third mode.

In order to choose an appropriate value of k, a computer program developed to evaluate frequency parameter Ω was run for different sets of the values of plate parameters for both the boundary conditions taking $k = 48, 49, \ldots, 56$. Then, the difference between the values of the frequency parameter Ω for two successive values of k was checked continually with varying values of k for various values of plate parameters till the accuracy of four decimals is attained i.e. $\left|\Omega_{k+1}^{(i)} - \Omega_{k}^{(i)}\right| \le 0.00005$ for all the three modes $i = 1, 2, 3$. For the present study, the number of terms k has been taken as 55 as there was no further improvement in the values of Ω for the first three modes for both the boundary conditions, considered here. The relative error $E_{rel} = \left|\Omega_j / \Omega_{56} - 1\right|$, $j = 48$ (1) 56 for a specified plate $N = 30$, $g = 5$, $\gamma = 0.5$, $K_f = 100$ has been shown in Figure 2 for the first three modes of vibration, as maximum deviations were obtained for this data.

The numerical results have been given in Tables 2-6 and presented in Figures 3-8. It is observed that the values of the frequency parameter Ω for simply supported plate are less than those for a clamped plate for the same set of the values of other parameters. The values of critical buckling loads for a clamped plate are higher than that for the corresponding simply supported plate.

g ↓	N ↓	$K_f = 0$ I	$K_f = 0$ II	$K_f = 0$ III	$K_f = 10$ I	$K_f = 10$ II	$K_f = 10$ III	$K_f = 100$ I	$K_f = 100$ II	$K_f = 100$ III
					$\gamma = -0.1$					
0	-10	*	24.6947	71.7593	*	24.8877	71.8258	7.8294	26.5617	72.4215
	0	5.2061	31.3465	78.0323	6.0573	31.4985	78.0934	11.0893	32.8352	78.6415
	10	9.3771	36.8133	83.8345	9.8756	36.9428	83.8914	13.5611	38.0882	84.4017
	20	12.1863	41.5633	89.2575	12.5741	41.6780	89.3109	15.6378	42.6964	89.7904
	30	14.4553	45.8201	94.3667	14.7839	45.9242	94.4173	17.4652	46.8503	94.8709
1	-10	*	15.6421	55.8186	*	15.9965	55.9185	7.5879	18.8902	56.8097
	0	4.3311	26.0778	64.9168	5.4740	26.2911	65.0027	11.4374	28.1379	65.7705
	10	9.4736	33.3985	72.8819	10.0485	33.5652	72.9584	14.2120	35.0295	73.6430
	20	12.6652	39.3732	80.0523	13.1012	39.5146	80.1219	16.5144	40.7656	80.7458
	30	15.1970	44.5482	86.6261	15.5623	44.6733	86.6904	18.5288	45.7836	87.2673
3	-10	*	10.7043	49.5148	*	11.2641	49.6378	7.5428	15.4141	50.7313
	0	4.0316	24.2748	60.4283	5.3379	24.5247	60.5291	11.7747	26.6694	61.4283
	10	9.6815	32.6303	69.6437	10.2953	32.8165	69.7311	14.7087	34.4474	70.5128
	20	13.0709	39.2351	77.7662	13.5321	39.3901	77.8445	17.1334	40.7587	78.5454
	30	15.7433	44.8718	85.1103	16.1284	45.0074	85.1818	19.2512	46.2100	85.8228
					$\gamma = 0.1$					
0	-10	*	23.1913	65.8277	*	23.4172	65.9077	8.4549	25.3598	66.6230
	0	4.6637	28.0774	70.2127	5.6750	28.2642	70.2877	11.2385	29.8922	70.9587
	10	8.7934	32.2196	74.3365	9.3695	32.3824	74.4073	13.4892	33.8122	75.0414
	20	11.5479	35.8821	78.2407	11.9926	36.0283	78.3080	15.4285	37.3186	78.9107
	30	13.7713	39.2018	81.9571	14.1464	39.3357	82.0214	17.1572	40.5207	82.5970
1	-10	*	15.8774	52.1002	*	16.2608	52.2182	8.5884	19.3729	53.2688
	0	3.8798	23.3582	58.4115	5.2225	23.6200	58.5168	11.7157	25.8569	59.4559
	10	8.9557	28.9379	64.0975	9.6146	29.1495	64.1935	14.2319	30.9889	65.0505
	20	12.0833	33.5969	69.3137	12.5797	33.7793	69.4024	16.3836	35.3787	70.1958
	30	14.5640	37.6824	74.1601	14.9785	37.8451	74.2431	18.2913	39.2794	74.9852
3	-10	*	12.3026	46.8449	*	12.8386	46.9883	8.8209	16.9156	48.2594
	0	3.6116	21.7432	54.3728	5.1373	22.0499	54.4963	12.1043	24.6390	55.5955
	10	9.1823	28.1317	60.9683	9.8833	28.3693	61.0784	14.7636	30.4242	62.0608
	20	12.5033	33.3097	66.9100	13.0272	33.5106	67.0104	17.0313	35.2672	67.9069
	30	15.1215	37.7831	72.3608	15.5576	37.9603	72.4536	19.0379	39.5198	73.2836

* denotes that the frequency parameter does not exist because of buckling

Table 2: Values of frequency parameter for simply supported plate.

g ↓	N ↓	$K_f = 0$			$K_f = 10$			$K_f = 100$		
		I	II	III	I	II	III	I	II	III
						$\gamma = -0.1$				
0	-10	7.5172	37.1300	88.5914	8.1346	37.2586	88.6453	12.3752	38.3967	89.1288
	0	11.0301	42.1337	93.9486	11.4597	42.2471	93.9994	14.7737	43.2544	94.4555
	10	13.6892	46.5857	99.0116	14.0375	46.6883	99.0598	16.8512	47.6019	99.4927
	20	15.9127	50.6374	103.8237	16.2132	50.7318	103.8697	18.7020	51.5739	104.2827
	30	17.8589	54.3807	108.4187	18.1272	54.4687	108.4627	20.3833	55.2538	108.8584
1	-10	2.9523	27.6389	70.4644	4.4740	27.8404	70.5435	11.0325	29.5921	71.2520
	0	9.1762	35.0520	78.1579	9.7721	35.2112	78.2293	14.0397	36.6128	78.8689
	10	12.6917	41.1082	85.1483	13.1288	41.2440	85.2139	16.5512	42.4471	85.8014
	20	15.4239	46.3613	91.5978	15.7854	46.4818	91.6587	18.7277	47.5527	92.2053
	30	17.7307	51.0646	97.6145	18.0460	51.1741	97.6717	20.6686	52.1488	98.1848
3	-10	*	23.5775	63.5922	1.9064	23.8349	63.6880	10.7123	26.0376	64.5436
	0	8.5418	32.6284	72.7539	9.2355	32.8151	72.8377	14.0101	34.4497	73.5871
	10	12.5274	39.5994	80.8687	13.0100	39.7534	80.9440	16.7390	41.1134	81.6192
	20	15.5146	45.4907	88.2278	15.9068	45.6249	88.2969	19.0769	46.8148	88.9163
	30	17.9994	50.6867	95.0084	18.3385	50.8071	95.0726	21.1466	51.8783	95.6482
						$\gamma = 0.1$				
0	-10	4.0304	33.3873	80.2309	5.1579	33.5443	80.2965	10.9472	34.9252	80.8841
	0	9.4027	37.3763	84.1680	9.9391	37.5164	84.2305	13.8624	38.7551	84.7907
	10	12.5785	40.9733	87.9263	12.9849	41.1011	87.9861	16.1893	42.2342	88.5224
	20	15.0473	44.2699	91.5272	15.3889	44.3882	91.5846	18.1766	45.4392	92.0999
	30	17.1308	47.3277	94.9881	17.4318	47.4384	95.0434	19.9374	48.4230	95.5400
1	-10	*	25.2393	64.3919	*	25.4816	64.4874	9.8148	27.5667	65.3403
	0	7.8223	31.0941	70.0212	8.5625	31.2908	70.1090	13.5079	33.0083	70.8939
	10	11.9213	35.9992	75.2246	12.4204	36.1691	75.3063	16.2360	37.6638	76.0373
	20	14.8481	40.2901	80.0827	15.2522	40.4420	80.1594	18.4959	41.7837	80.8464
	30	17.2396	44.1450	84.6537	17.5891	44.2836	84.7262	20.4680	45.5121	85.3764
3	-10	*	21.8582	58.4972	*	22.1633	58.6120	9.4880	24.7409	59.6350
	0	7.2815	28.9442	65.1798	8.1404	29.1748	65.2828	13.6190	31.1735	66.2021
	10	11.8744	34.5974	71.2300	12.4207	34.7904	71.3241	16.5450	36.4814	72.1662
	20	15.0272	39.4172	76.7929	15.4631	39.5867	76.8803	18.9401	41.0803	77.6619
	30	17.5729	43.6803	81.9666	17.9474	43.8333	82.0484	21.0199	45.1868	82.7812

* denotes that the frequency parameter does not exist because of buckling

Table 3: Values of frequency parameter for clamped plate.

γ ↓	g ↓	Modes ↓	$K_f = 0$	$K_f = 10$	$K_f = 100$	$K_f = 0$	$K_f = 10$	$K_f = 100$
			Simply supported plate			Clamped plate		
-0.3	1	I	2.8882	4.1840	12.8866	12.5432	13.0929	17.7154
		II	12.3006	12.4772	16.8314	21.6535	22.5547	26.7538
		III	30.1499	30.2442	31.2944	56.3859	56.5335	58.1104
	3	I	2.2913	3.5856	10.7358	11.9509	13.4998	16.0191
		II	9.7585	9.9363	15.7235	16.7643	17.8642	19.9876
		III	23.9189	24.0135	25.1867	44.7328	44.8811	46.5370
0.3	1	I	2.0116	4.3952	23.5744	5.1932	6.8207	21.1783
		II	18.6339	18.9851	24.4055	29.5455	30.1018	35.3628
		III	55.4928	55.6702	56.5676	74.8228	75.1151	77.7286
	3	I	1.5959	3.9919	12.7556	4.1199	5.7469	19.9314
		II	14.7829	15.1233	19.4774	23.4395	23.9961	29.4242
		III	44.0243	44.2004	44.7283	59.3595	59.6517	62.2554

Table 4: Critical buckling load in compression.

In Figures 3(a, b, c), the behaviour of volume fraction index g on the frequency parameter Ω for two values of in-plane force parameter $N =$ -5, 10 and taper parameter $\gamma =$ -0.5, 0.5 for a fixed value of foundation parameter $K_f = 50$ for both the plates has been presented. It has been observed that the value of frequency parameter Ω decreases with the increasing value of g for both tensile as well as compressive in-plane forces for both the plates except for the fundamental mode of vibration. The corresponding rate of decrease is higher for smaller values of $g\,(< 2)$ as compared to the higher values of g (> 3). Further, it increases with the increase in the number of modes. In case of fundamental mode (Figure 3(a)), the frequency parameter Ω increases as g increases for both the plates when plate becomes thinner and thinner towards the boundary (i.e. $\gamma = 0.5$) in presence of tensile in-plane force (i.e. $N = 10$). However, for the simply supported plate vibrating in the fundamental mode, the frequency parameter is found monotonically increasing with increasing values of g irrespective of the values of γ (± 0.5) as well as in-plane force parameter N (-5, 10).

The effect of in-plane force parameter N on the frequency parameter Ω for three different values of taper parameter $\gamma =$ -0.3, 0, 0.3 taking $g = 5$ and $K_f = 10$ for both the plates vibrating in the first three modes has been shown in Figures 4(a, b, c). It has been noticed that the value of the frequency parameter increases with the increasing value of N whatever be the values of other plate parameters. This effect increases with the increasing number of modes. The rate of increase in the values of Ω with N increases as the plate becomes thicker and thicker towards the outer edge i.e. γ changes from positive values to negative values. The rate of increase is higher for simply supported plate as compared to clamped plate.

Figures 5(a, b, c) show the graphs for foundation parameter K_f verses frequency parameter Ω for two different values of taper parameter γ = -0.5, 0.5 and in-plane force parameter N = -5, 10 for a fixed value of volume fraction index g = 5 for all the three modes. It can be seen that the frequency parameter Ω increases with the increasing values of foundation parameter K_f. This effect is more pronounced for tensile in-plane forces (i.e. N = 10) as compared to compressive in-plane forces (i.e. N = -5) with respect to the change in the values of taper parameter γ from 0.5 to -0.5 keeping other plate parameters fixed and also increases with the increase in the number of modes for both the plates.

To study the effect of taper parameter γ on the frequency parameter Ω, the graphs have been plotted for fixed value of foundation parameter K_f = 50 for two values of volume fraction index g = 0, 5 and three values of in-plane force parameter N = -5, 0, 10 and given in Figures 6(a, b, c) for all the three modes and for both the plates. It has been observed that for a clamped plate vibrating in the fundamental mode of vibration, the values of the frequency parameter Ω decrease monotonically with the increasing values of taper parameter γ from -0.5 to 0.5 whatever be the values of g as well as N. In case of simply supported plate, for g = 0, the values of frequency parameter has a point of minima having a tendency of shifting from -0.1 towards 0.3 as N changes from compressive to tensile i.e. takes the values as -5, 0, 10. However, for g = 5, the values of Ω increase continuously for N = -5, 0 while there is a point of minima in the vicinity of γ = 0 for N = 10. For the second mode of vibration, the values of frequency parameter Ω monotonically decrease with the increasing values of g and N for both the plates except for simply supported plate for N = -5, g = 5. In this case, there is a point of maxima in the vicinity of γ = -0.1. The rate of decrease in the values of Ω with increasing values of taper parameter γ is higher in case of clamped plate as compared to simply supported plate keeping other parameters fixed. The effect of in-plane force parameter is more pronounced for γ = 0.5 for both the plates whatever be the value of g. In case of the third mode of vibration, the behaviour of the frequency parameter Ω with the taper parameter γ is almost similar to that of the second mode for both the plates except that the rate of decrease is much higher than the second mode.

By allowing the frequency to approach zero, the values of the critical buckling load parameter N_{cr} in compression for different values of volume fraction index g = 1, 3; taper parameter γ = -0.3, 0.3 and foundation parameter K_f = 0, 10, 100 for both the plates have been computed in Table 4. For selected values of volume fraction index g = 0, 5; taper parameter γ = -0.5, -0.1, 0, 0.1, 0.5 for K_f = 50, the plots for the critical buckling load parameter N_{cr} for both the plates vibrating in the fundamental mode of vibration have been given in Figures 7(a, b). From the results, it has been observed that the values of the critical buckling load parameter N_{cr} for a clamped plate are higher than that for the corresponding simply supported plate. Also, the value of N_{cr} decreases with the increasing values of the volume fraction index g for both the boundary conditions whereas the value of N_{cr} increases with the increasing values of K_f for all the values of g and γ for both the plates. A comparative study for critical buckling load parameter N_{cr} for an isotropic plate with Gupta and Ansari (1998) obtained by using Ritz method, Vol'mir (1966) exact solutions, Pardoen (1978) obtained by using finite element method have been presented in Table 5. An excellent agreement among the results has been noticed. Three-dimensional mode shapes for the first three modes of vibrations for a specified plate $g = 5$, $N = 30$, $\gamma = -0.5$, $K_f = 100$ are shown in Figure 8 for both the plates.

The results for the frequency parameter Ω with those obtained by Rayleigh-Ritz method (Singh and Saxena, 1995), exact element method (Eisenberger and Jabareen, 2001), generalized differential quadrature rule (Wu and Liu, 2001) has been given in Table 6. A close agreement of the results for both the boundary conditions shows the versatility of the present technique.

Boundary conditions	Refs.	I	II	III
		$\gamma = -0.5$		
Simply Supported	Present	6.2927	37.7423	93.0342
	[31][a]	6.2928	37.743	93.042
Clamped	Present	14.3021	51.3480	112.6360
	[31][a]	14.302	51.349	112.64
		$\gamma = -0.3$		
Simply Supported	Present	5.7483	34.5625	85.6206
	[31][a]	5.7483	34.563	85.623
Clamped	Present	12.6631	46.7813	103.4123
	[31][a]	12.663	46.782	103.41
		$\gamma = -0.1$		
Simply Supported	Present	5.2061	31.3465	78.0323
	[31][a]	5.2061	31.346	78.032
Clamped	Present	11.0301	42.1337	93.9486
	[31][a]	11.030	42.134	93.949
		$\gamma = 0$		
Simply Supported	Present	4.9351	29.7200	74.1561
	[31][a]	4.9351	29.720	74.156
	[33][c]	4.935	29.720	74.156
Clamped	Present	10.2158	39.7711	89.1041
	[31][a]	10.216	39.771	89.104
	[33][c]	10.216	39.771	89.104
		$\gamma = 0.1$		
Simply Supported	Present	4.6637	28.0774	70.2127
	[31][a]	4.6637	28.077	70.213
Clamped	Present	9.4027	37.3763	84.1680
	[31][a]	9.4027	37.376	84.168
		$\gamma = 0.3$		

Boundary conditions (cont.)	Refs. (cont.)	I (cont.)	II (cont.)	III (cont.)
Simply Supported	Present	4.1158	24.7265	62.0704
	[32][b]	4.11575858	24.72653378	62.07039882
	[33][c]	4.116	24.727	62.071
	[31][a]	4.1158	24.727	62.071
Clamped	Present	7.7783	32.4610	73.9467
	[32][b]	7.77831060	32.46099735	73.94674509
	[33][c]	7.779	32.462	73.948
	[31][a]	7.7783	32.461	73.947
		$\gamma = 0.5$		
Simply supported	Present	3.5498	21.2386	53.4402
	[31][a]	3.5498	21.239	53.441
Clamped	Present	6.1504	27.3002	63.0609
	[31][a]	6.1504	27.300	63.062

[a] Singh and Saxena (1995) by Rayleigh-Ritz method
[b] Eisenberger and Jabareen (2001) by exact element method
[c] Wu and Liu (2001) by generalized differential quadrature rule

Table 5: Comparison of frequency parameter Ω for linear varying thickness taking $N = 0$, $g = 0$, $K_f = 0$.

Boundary condition	Ref.	First mode	Second mode	Third mode
Simply Supported	Present	4.1978	29.0452	73.4768
	Gupta and Ansari [43]	4.1978	29.0452	73.4768
	Vol'mir [44]	4.1978	29.0452	73.4768
	Pardoen [45]	4.1978	29.0495	73.5495
Clamped	Present	14.6820	49.2185	103.4995
	Gupta [43]	14.6820	49.2158	103.4995
	Vol'mir [44]	14.6820	49.2158	103.4995
	Pardoen [45]	14.6825	49.2394	103.7035

Table 6: Comparison of critical buckling load parameter N_{cr} for isotropic plate ($g = 0, \gamma = 0, K_f = 0$).

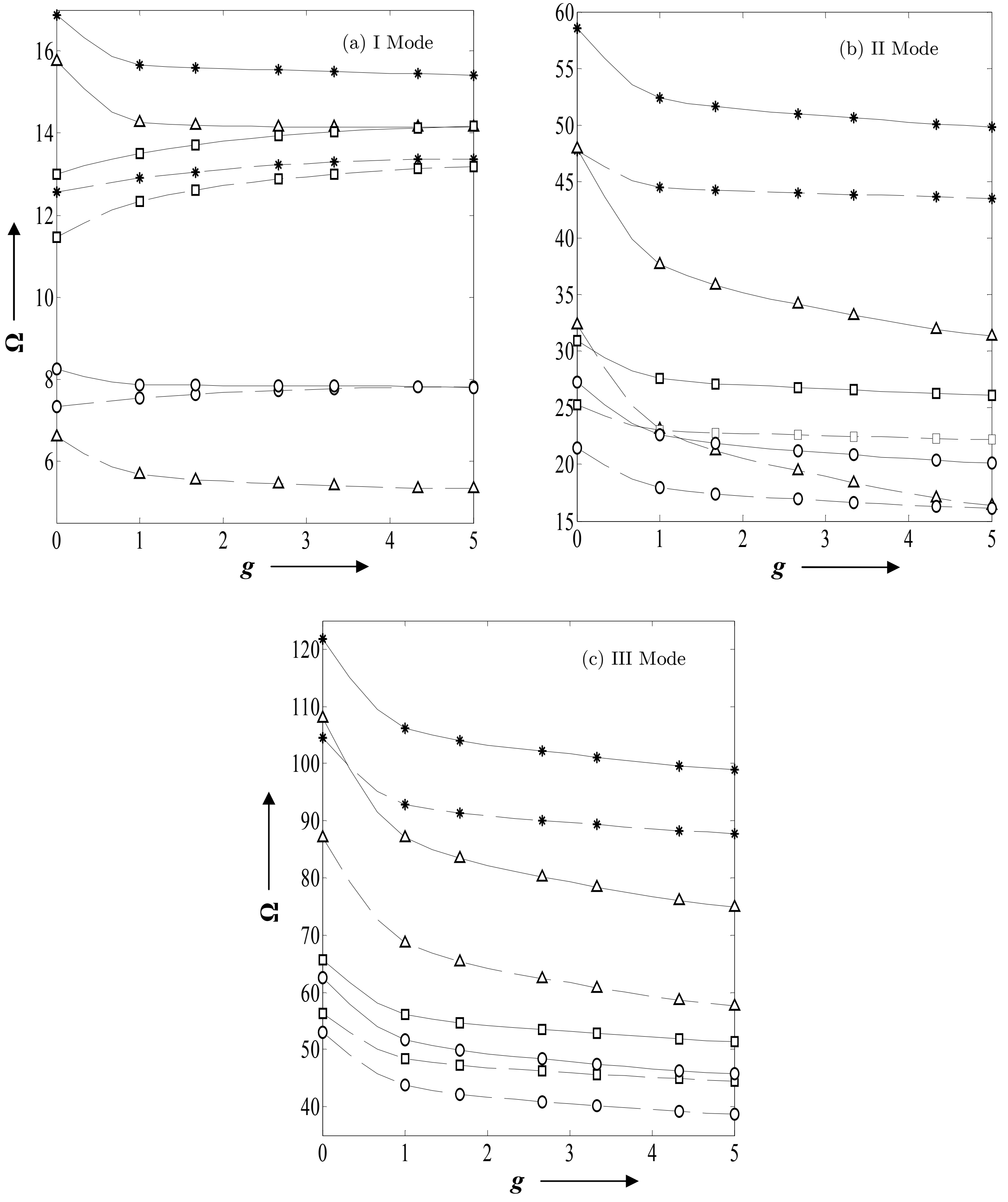


Figure 3: Frequency parameter Ω for _ _ _ _ _ _, simply supported plate; ________, clamped plate. Δ, γ = -0.5, N = -5; ✱ , γ = -0.5, N = 10; ○, γ = 0.5, N = -5; □, γ = 0.5, N = 10; $K_f = 50$.

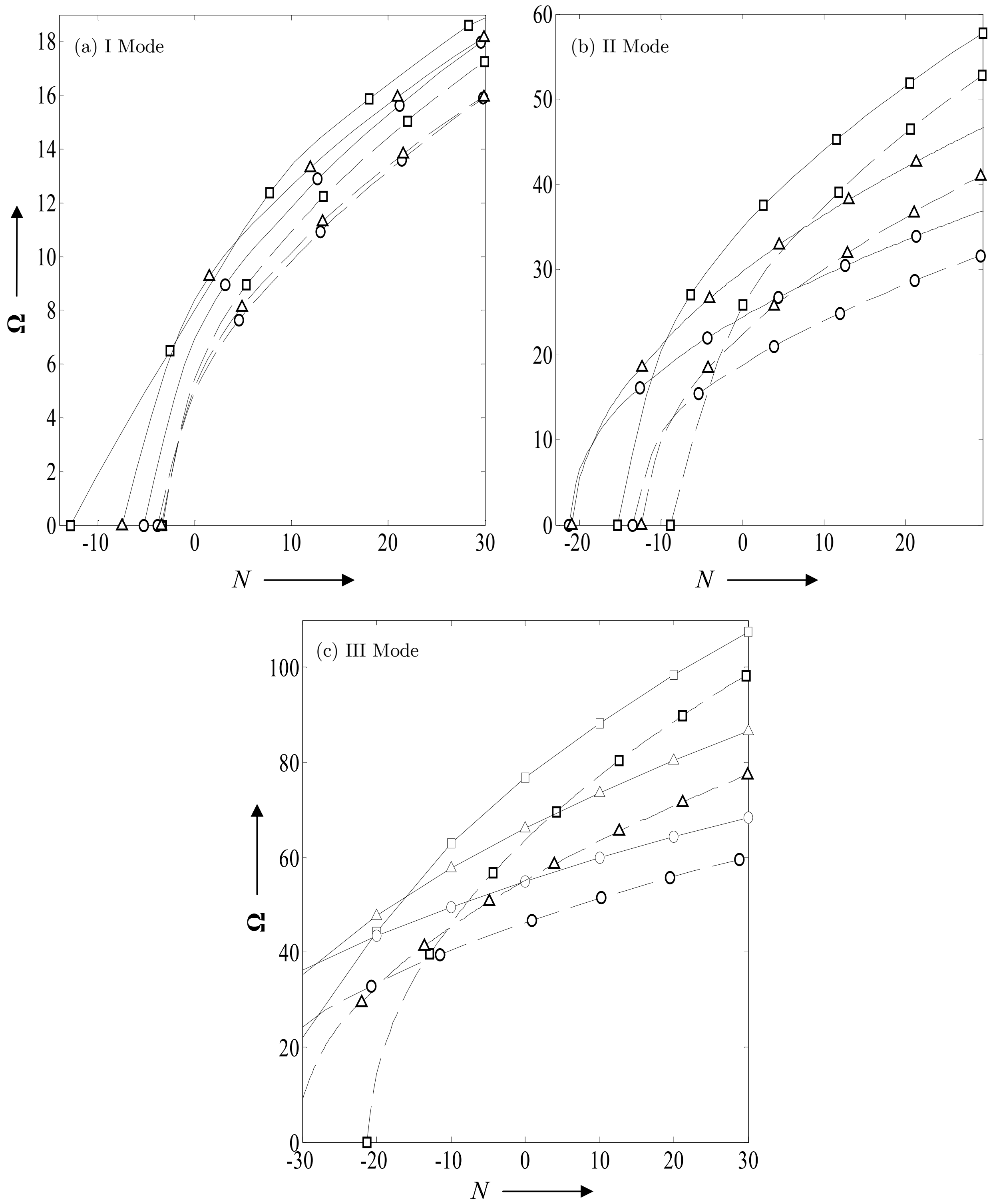


Figure 4: Frequency parameter Ω for _ _ _ _ _ _, simply supported plate; ________, clamped plate. □, γ = -0.3; Δ, $\gamma = 0$; ○, $\gamma = 0.3$; $g = 5$; $K_f = 10$.

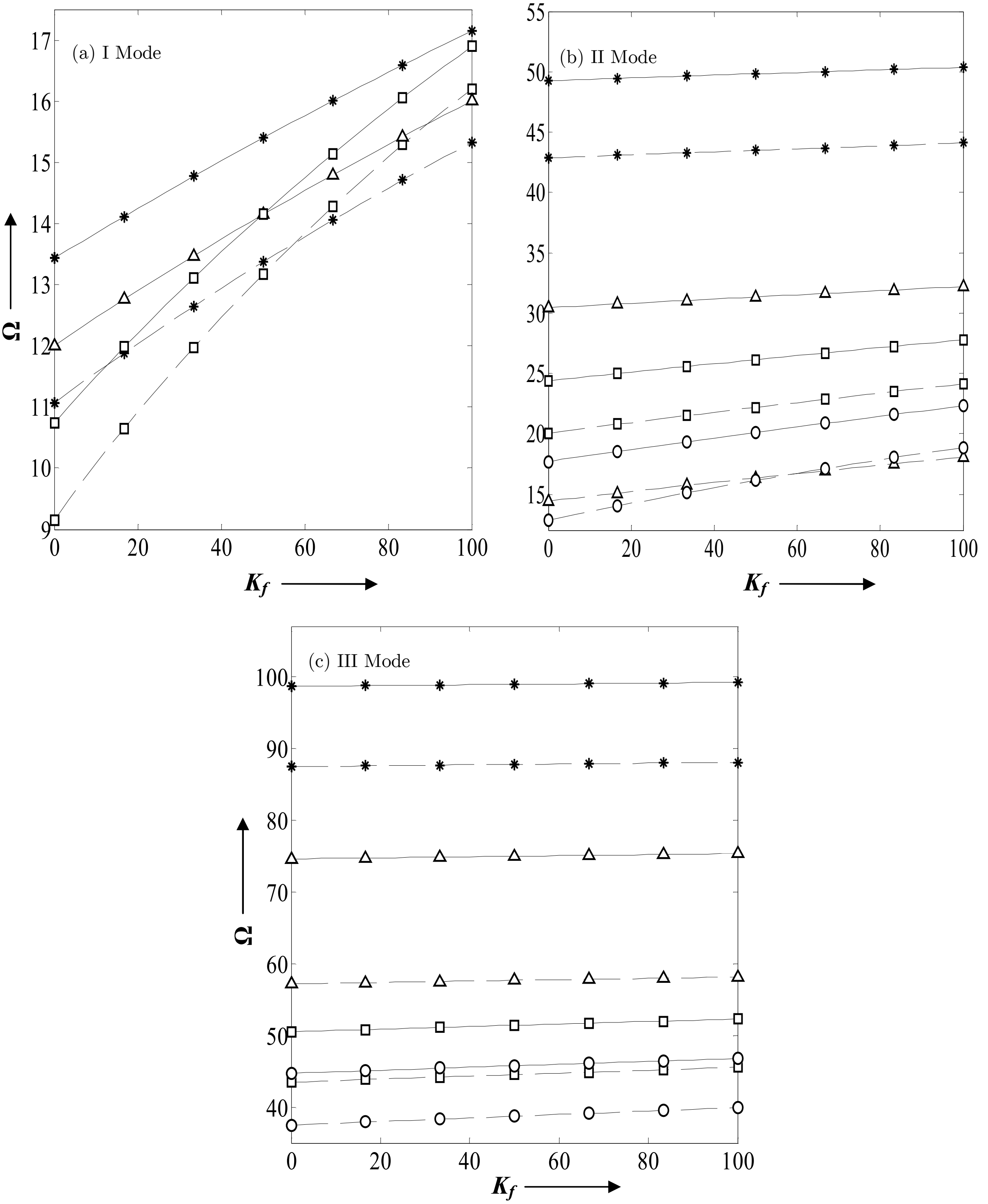


Figure 5: Frequency parameter Ω for _ _ _ _ _, simply supported plate; _______, clamped plate. Δ, γ = -0.5, N = -5; ✱ , γ = -0.5, N = 10; ○ , γ = 0.5, N = -5; □, γ = 0.5, N = 10; $g = 5$.

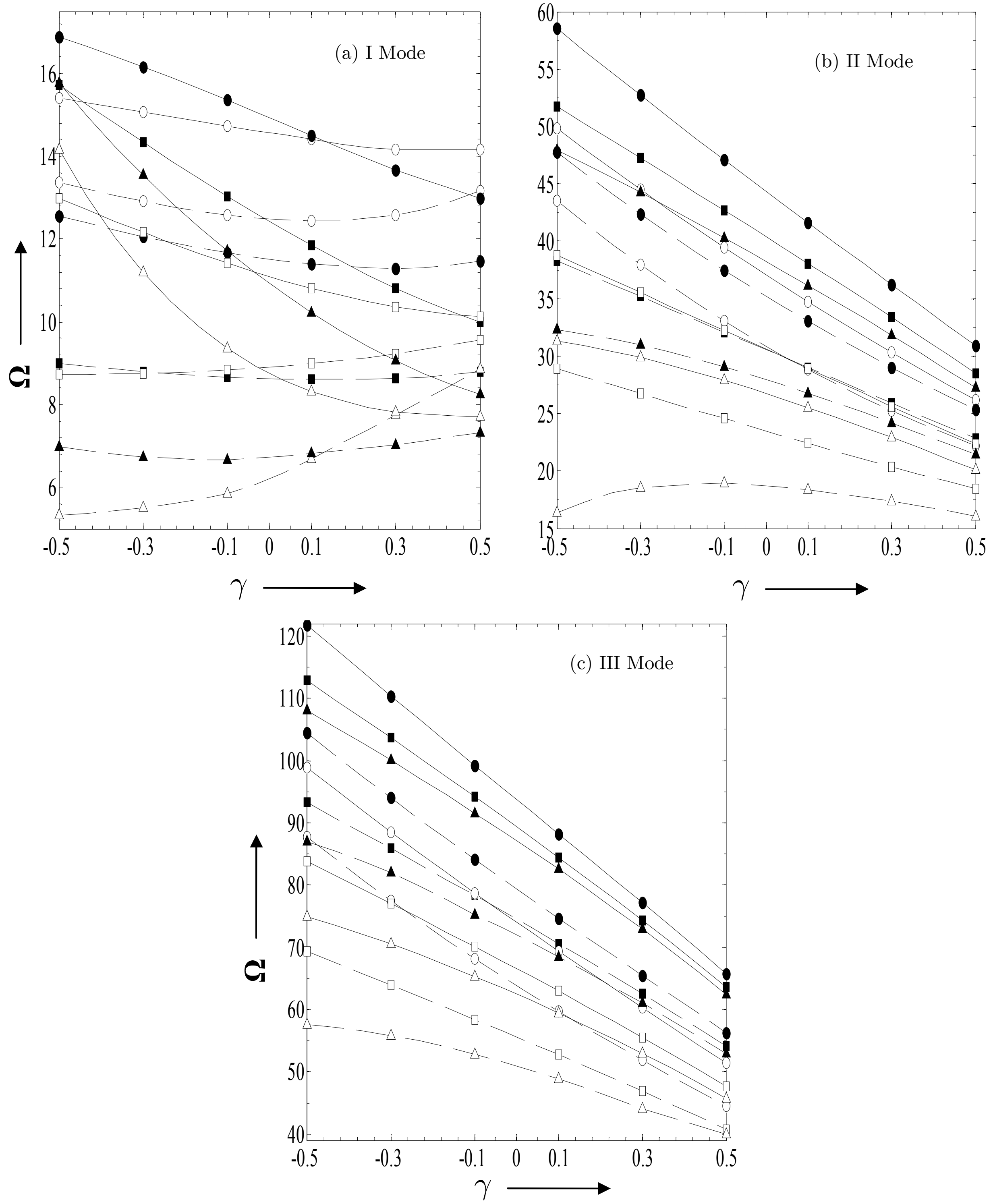
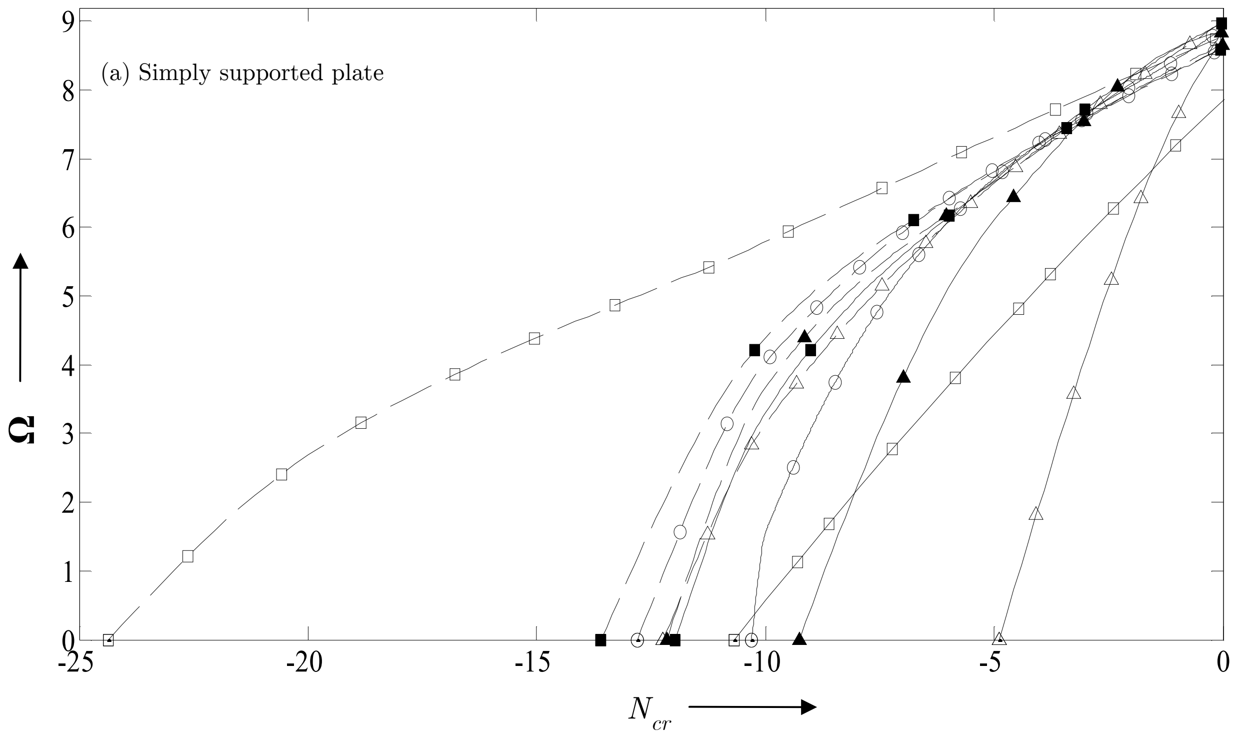


Figure 6: Frequency parameter Ω for _ _ _ _ _ _, simply supported plate; ________, clamped plate. ▲, N = -5, $g = 0$; ■, $N = 0$, $g = 0$; ●, $N = 10$, $g = 0$; Δ, N = -5, $g = 5$; □, $N = 0$, $g = 5$; ○, $N = 10$, $g = 5$; $K_f = 50$.



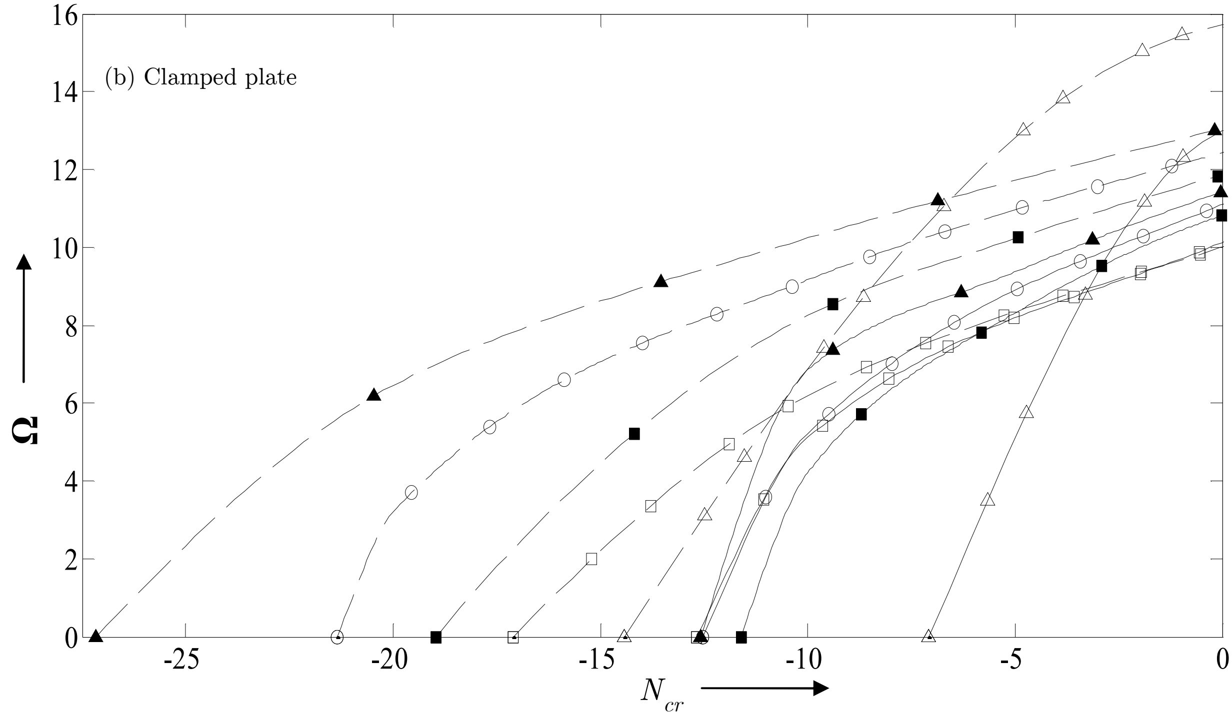


Figure 7: Critical buckling load parameter N_{cr} for ________, $g = 5$; _ _ _ _ _ _, $g = 0$; Δ, γ = -0.5 ;▲, γ = -0.1; ○, γ = 0 ; ■, γ = 0.1; □, γ = 0.5; $K_f = 50$.

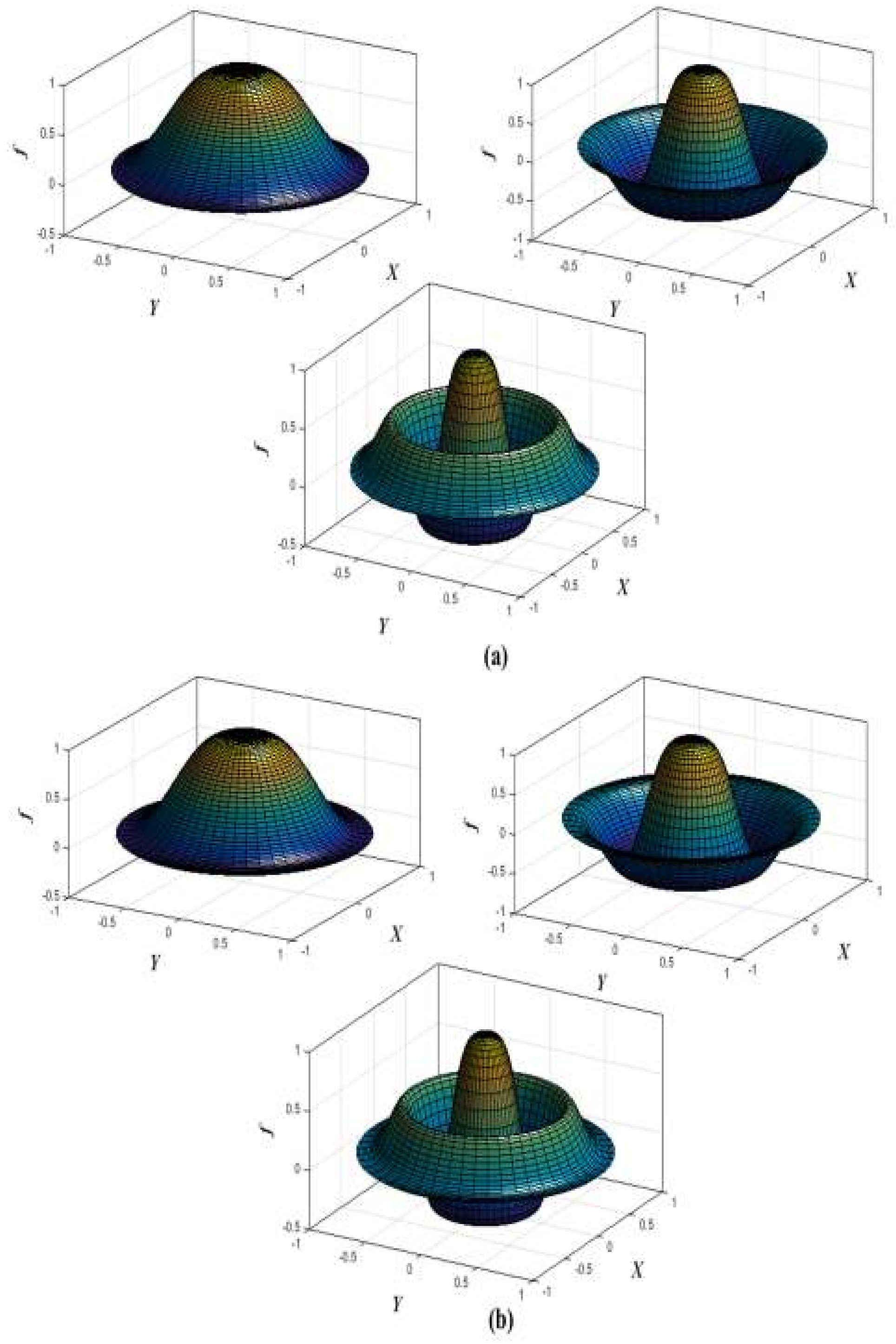


Figure 8: Three dimensional mode shapes for (a) Simply supported plate (b) Clamped plate for $g = 5$, $N = 30$, $\gamma = -0.5$, $K_f = 100$.

g	Modes	γ = -0.5		γ = 0.5		γ = -0.5		γ = 0.5	
		$N = 0$	$N = 30$	$N = 0$	$N = 30$	$N = 0$	$N = 30$	$N = 0$	$N = 30$
		Simply supported plate				Clamped plate			
0	I	4.29	12.19	-1.84	-3.03	26.58	7.83	19.52	4.73
	II	25.32	44.36	25.27	31.47	28.11	36.56	29.38	31.62
	III	25.19	39.82	27.38	31.56	26.22	35.27	28.83	31.67
5	I	-1.99	11.36	-7.52	-13.08	17.04	6.84	8.77	-6.91
	II	23.24	48.94	21.42	33.52	26.8	40.76	26.89	32.22
	III	24.82	47.86	26.67	34.52	25.95	41.58	28.3	33.56

Table 7: Percentage variation in the values of frequency parameter with respect to $\gamma = 0$ when γ changes from -0.5 to 0 and 0 to 0.5 for $N = 0$, 30 and $g = 0$, 5 taking $K_f = 50$.

6 CONCLUSIONS

The effect of thickness variation and elastic foundation on the buckling and free axisymmetric vibration of functionally graded circular plate has been analysed employing differential transform method. The numerical results show that:

1. Values of frequency parameter Ω for an isotropic plate are higher than those for the corresponding FGM plate i.e. frequency parameter decreases as the contribution of metal constituent increases.

2. The values of frequency parameter Ω for a clamped plate are higher than that for the corresponding simply supported plate whatever be the values of other plate parameters.

3. As the plate becomes thicker and thicker towards the outer edge i.e. the value of the taper parameter γ changes from negative to positive, the values of the frequency parameter Ω decreases continuously and the rate of decrease is higher for the clamped plate as compared to the simply supported plate.

4. With the increase in the value of volume fraction index g, the frequency parameter Ω decreases irrespective to the values of the other plate parameters. The rate of decrease increases with increase in the number of modes for both the plates.

5. The values of frequency parameter also increase with the increase in the values of foundation parameter K_f whatever be the values of other plate parameters.

6. The values of critical buckling loads for an isotropic plate ($g = 0$) are higher than the corresponding FGM plate ($g > 0$) i.e. in order to obtain the highest critical buckling loads, the ceramic isotropic plate is much suitable than FGM plate.

7. The clamped plate pursue more critical buckling load than the corresponding simply supported plate.

8. The values of the critical buckling load parameter N_{cr} decreases with the increasing values of volume fraction index g keeping other plate parameters fixed.

9. As the foundation stiffness increases, the values of critical buckling load parameter N_{cr} increases for both the plates.

10. The percentage variations in the values of the frequency parameter Ω with varying values of in-plane force parameter $N = 0$, 30 and volume fraction index $g = 0$, 5 for both the plates for two cases (i) when the plate becomes thicker and thicker toward the outer edge i.e. γ changes from 0 to -0.5 (ii) when the plate becomes thinner and thinner towards the outer edge i.e. γ change from 0 to 0.5 for all the three modes have been computed and given in Table 7. It has been noticed that for both the plates, in absence of in-plane force when the plate changes its nature from isotropic to FGM, the percentage variation decreases for all the modes and for both the cases. This behaviour remains same in the presence of tensile in-plane force, when the plate is vibrating in the fundamental mode of vibration while percentage variation increases for the second and third modes of vibration.

11. The differential transform method, being straightforward and easy to apply for such type of problems, gives highly accurate results with less computational efforts as compared to the other conventional methods like differential quadrature and finite element methods etc. The results presented in this paper can serve as benchmark solutions for future investigations.

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