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Determination of the Appropriate Gradient Elasticity Theory for Bending Analysis of Nano-beams by Considering Boundary Conditions Effect

Abstract

In the present paper, a critique study on some models available in the literature for bending analysis of nano-beams using the gradient elasticity theory is accomplished. In nonlocal elasticity models of nano-beams, the size effect has not been properly considered in governing equations and boundary conditions. It means that in these models, because of replacing of the size effect with the inertia gradient effect, the size dependency has been ignored in bending analysis of nano-beams. Therefore, as the beam dimensions increase in comparison to its material length scale parameter, the obtained solution based on the gradient elasticity theory (either in the nonlocal elasticity theory or the strain gradient elasticity theory) should converge to the classical elasticity solution. Hence, satisfying of boundary conditions is a crucial point. In this paper, governing equations and boundary conditions are presented based on two gradient elasticity theories (i.e., nonlocal elasticity and strain gradient elasticity theories). Also, boundary conditions in strain gradient elasticity theory are modified based on a dimensional analysis approach. The results indicate that the strain gradient elasticity theory captures the size effect more sensitive in comparison with the nonlocal elasticity theory in bending analysis. In addition, modified boundary conditions in strain gradient elasticity theory can lead to converge the classical solution at large scales. To prove that the boundary conditions of nano-beam have the direct effect on mechanical behavior of structure, the size-dependent Young modulus of carbon nanotube (CNT) is investigated and the results show that the prediction of strain gradient elasticity theory with modified boundary conditions is in a good agreement with experimental results.

Keywords

Nonlocal elasticity theory; strain gradient elasticity theory; nano-beams; size effect; boundary conditions.

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1 INTRODUCTION

In the last two decades, the interest in the use of very small scale structures for several purposes has been increased. Micro-electromechanical systems (MEMS), nano-electromechanical systems (NEMS), biosensors, actuators and nanocomposites are some examples of the applications of such small scale structures. The nano-beam (NB) as one of the structures that are used in NEMS has attracted the attention of many researchers (Mahmoud et al., 2013). It can also be employed as an element for mechanical modeling of nanotubes (Wang and Hu, 2005) which are used in polymer nanocomposites extensively (Bal and Samal, 2007).

To determine the mechanical behavior of nanostructures, as well as the experimental methods and atomistic simulations, the continuum mechanics approach is also available. This approach is computationally less expensive than the former approaches and its formulation is simple. So, it can be employed as an alternative way to simulate the mechanical behavior of nanostructures (Arash and Wang, 2012).

Experimental evidences show that when the dimensions of the continuum and material length scale parameter are at the same order, the size effect cannot be neglected (Tang and Alici, 2011a, 2011b). Due to the lack of existing such intrinsic length scales in classical continuum theory, these experimental observations cannot be captured by this theory. Hence, various generalized continuum theories, including couple stress (Mindlin and Tiersten, 1962; Toupin, 1962; Koiter, 1964), micropolar and micro-morphic (Eringen, 1976, 1999), nonlocal (Eringen, 1976, 1983; Eringen and Wegner, 2003) and strain gradient (Mindlin, 1964, 1965; Toupin, 1964) theories have been employed and further developed. Among the aforementioned theories, the nonlocal and strain gradient theories are used extensively. Generally, these two theories can be considered as the gradient elasticity theories, in which in addition to strain or stress, their gradients are taken into account. In nonlocal continuum field theories, the material behavior at a point is influenced not only by that point, but also by the state of all points of the body. The nonlocal elasticity theory was initiated by Eringen (1972, 1983; Eringen and Wegner, 2003) and Eringen and Edelen (1972). In recent decades, many researchers have used the nonlocal elasticity theory to analyze mechanical behaviors of micro and nano-beams in static loading conditions (Peddieson et al., 2003; Reddy, 2007; Wang and Liew, 2007; Kiani, 2010a, 2010b; Janghorban, 2012).

In the strain gradient elasticity theory, in addition to the strain, gradients of strain must be considered in the strain energy density function of the deformable body. The early investigations on the strain gradient elasticity theory can be found in studies of Mindlin (1964) and Kröner (1963). After 1960s, many studies on the Mindlin's general strain gradient elasticity theory were carried out and several theories such as simplified strain gradient theory (Aifantis, 1992), modified couple stress theory (Yang et al., 2002) and modified strain gradient theory (Lam et al., 2003) have been developed.

In recent years, many works have been published on the analysis of an Euler-Bernoulli microbeam based on the simplified strain gradient elasticity (Lazopoulos and Lazopoulos, 2010; Rajabi and Ramezani, 2011; Lazopoulos, 2013), the modified strain gradient elasticity (Akgöz and Civalek, 2011) and modified couple stress theories (Park and Gao, 2006; Akgöz and Civalek, 2011). Furthermore, analysis of the Timoshenko beam model using the modified couple stress theory (Ma et al., 2008; Gao, 2014) and the modified strain gradient elasticity (Wang et al., 2010; Asghari et al., 2012; Kahrobaiyan et al., 2014) has been investigated and the Reddy-Levinson beam theory based on modified couple stress theory (Ma et al., 2010) has been studied.

A comprehensive survey on the formulation and governing equations of previous works shows some contradictions and discrepancies. As the first example, in a paper by Wang and Liew (2007), it seems that the nonlocal effect depends on the position of applied concentrated load on the nanobeam. While, in another paper (Peddieson et al., 2003) it has been stated that the governing equations of each beam segment which is not acted upon by a distributed load has the same local or classical governing equation. Also, in a paper by Lazopoulos and Lazopoulos (2010) there is a contradiction on boundary conditions which deviates the physical interpretation. For instance, based on their results, when the dimensions of a simply supported nano-beam are large enough, the displacement field does not converge to the classical response of the beam. The main idea of the present work is to answer to two basic questions: 1) which one of the gradient elasticity theories (nonlocal elasticity of strain gradient elasticity) can correctly capture the size effect in bending analysis of nano-beams? 2) what is the basic criterion in choosing the additional non-classical boundary conditions of the strain gradient elasticity theory?

In the present research, three different formulations (a nonlocal elasticity formulation and two strain gradient elasticity formulations) are presented for bending analysis of nano-beams. Firstly, the general governing equations and corresponding boundary conditions of each formulation are mentioned. Then, the general formulations are simplified for two well-known case studies, i.e., the cantilever Euler-Bernoulli beam and the simply-supported Euler-Bernoulli beam. A comparison between the nonlocal, strain gradient and classical beam theories formulations will be performed to clarify the ability of each formulation to capture the size effect. Using dimensional analysis, the importance of several terms in governing equations and boundary conditions in strain gradient formulations either in small or large scales will be cleared and this emphasizes the effect of boundary conditions on the exact response of nano-beams. Also, the size-dependent Young's modulus of the CNT is investigated based on the prediction of two formulated strain gradient elasticity theories.

2 MATERIAL AND METHODS

In this section, three formulations of the gradient elasticity theory in bending analysis of Euler-Bernoulli nano-beams are presented. Then, based on these formulations the bending problems of cantilever and simply-supported nano-beams are investigated. This can provide a basis to compare the ability of different gradient elasticity theories in static analysis of nano-beams. Also, by employing the dimensional analysis (see the Appendix), the significance of different terms, especially those in boundary conditions will be cleared. Furthermore, proper boundary conditions can be chosen easily to construct a physical meaningful interpretation.

2.1 Nonlocal nano-beams formulation (Formulation I)

In this subsection, a work of Reddy (2007) which has formulated the nonlocal beams using various theories of bending is considered. Here, the Euler-Bernoulli beam formulation is considered. According to his formulation, the nonlocal beam governing equation without an axial motion is:

$$\frac{\partial^2 M}{\partial x^2} + q(x) = \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^2}{\partial x^2}(\frac{\partial^2 w}{\partial t^2}) \tag{1}$$

where *M, I, ϱ, A, w(x,t), q(x)* are the nonlocal bending moment, the second moment of cross section area, the mass density, the cross section area, the transverse displacement and the distributed load, respectively. The general form of the nonlocal constitutive equation is as follows (Eringen, 1983; Lu et al., 2007; Reddy, 2007):

$$\sigma_{ij}^n - g^2 \sigma_{ij,mm}^n = C_{ijkl} u_{k,l} \tag{2}$$

where σ_{ij}^n, g, C_{ijkl} and u_k are the nonlocal stress, material length scale parameter, stiffness elasticity tensor and displacement field, respectively. Since the only nonzero stress component in the Euler-Bernoulli beam theory is σ_{xx}, Eq. (2) can be simplified as below:

$$\sigma_{xx} - g^2 \frac{d^2 \sigma_{xx}}{dx^2} = E\varepsilon_{xx} \tag{3}$$

where E and ε_{xx} are the Young's modulus and strain component, respectively. If we consider Eq. (3), a special form of the nonlocal stress resultant can be extracted. To this end, firstly both sides of Eq. (3) are multiplied to z and the resulting expression is integrated through the volume. In this way, the nonlocal stress resultant can be expressed as below:

$$M - g^2 \frac{\partial^2 M}{\partial x^2} = -EI \frac{\partial^2 w}{\partial x^2} \tag{4}$$

where M and ε_{xx} are defined as below:

$$M = \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{xx}\, dA \quad , \quad \varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2} \tag{5}$$

To obtain the displacement and stress resultant, Eqs (1) and (4) must be solved simultaneously. But, by combining them, one can obtain the governing equation in terms of the displacement. If we replace the second derivative of stress resultant from Eq. (1) into Eq. (4) and then apply the second derivative with respect to x on both sides and again substitute the resulting expression into Eq. (1) , the following governing equation will be obtained:

$$\frac{\partial^2}{\partial x^2}(-EI \frac{\partial^2 w}{\partial x^2}) + (1 - g^2 \frac{\partial^2}{\partial x^2}) q(x) = (1 - g^2 \frac{\partial^2}{\partial x^2})(\rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^2}{\partial t^2}(\frac{\partial^2 w}{\partial x^2})) \tag{6}$$

Also, in the aforementioned paper (Reddy, 2007) the essential and natural boundary conditions at both ends of a beam resulted from the calculus of variation have been presented as:

$$\bar{V}+\frac{\partial}{\partial x}(EI\frac{\partial^2 w}{\partial x^2})-g^2\frac{\partial}{\partial x}\left[-q(x)+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\frac{\partial^2}{\partial t^2}(\frac{\partial^2 w}{\partial x^2})\right]-\rho I\frac{\partial}{\partial t}(\frac{\partial w}{\partial x})=0 \quad or \quad \delta w=0$$

$$-\bar{M}-EI\frac{\partial^2 w}{\partial x^2}+g^2\left[-q(x)+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\frac{\partial^2}{\partial t^2}(\frac{\partial^2 w}{\partial x^2})\right]=0 \quad or \quad \delta(\frac{\partial w}{\partial x})=0 \tag{7}$$

where $\bar{V}$ and $\bar{M}$ denote the prescribed shear force and bending moment at the boundaries, respectively and δ indicates the variational operator. Inserting g=0 in Eq. (4) leads to the classical stress resultant relation. The second derivative of the stress resultant in Eq. (4) indicates the nonlocal effect contribution. By considering Eq. (1) it should be noted that the nonlocal effect contribution will be replaced by the inertia and distributed load terms as follows:

$$g^2\frac{\partial^2 M}{\partial x^2}=g^2(\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\frac{\partial^2}{\partial t^2}(\frac{\partial^2 w}{\partial x^2}))-g^2 q(x) \tag{8}$$

To solve a static problem, the inertia terms in Eqs. (6) and (7) should be vanished. In following this will be done in both cases of simply-supported and cantilever beams.

2.1.1 Case study I: Simply-supported beam

For a simply-supported beam subjected to a constant distributed load as shown in Figure 1:, the governing equation and boundary conditions will be obtained as follows:

$$\frac{\partial^2}{\partial x^2}(-EI\frac{\partial^2 w}{\partial x^2})+q=0$$

$$w(0)=w(L)=0 \tag{9}$$

$$\bar{M}(0)=\bar{M}(L)=0 \quad , \quad \bar{M}=-EI\frac{\partial^2 w}{\partial x^2}-g^2 q$$

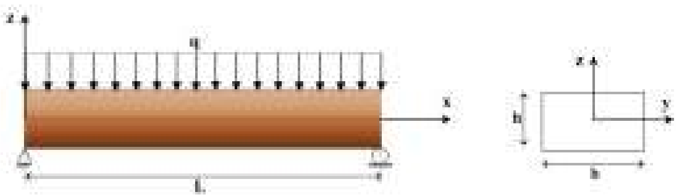

Figure 1: The simply-supported nano-beam subjected to a constant distributed load.

Solving of the governing equation yields:

$$w = \frac{1}{6}C1x^3 + \frac{1}{2}C2x^2 + C3x + C4 + \frac{1}{24}\frac{qx^4}{EI} \tag{10}$$

where coefficients C_i, $i = 1,2,...,4$ are the integration constants which can be computed by applying boundary conditions. Hereafter, we should emphasize that it is difficult to report all the integration constants here. Some of them are so long and we used the Maple computerized algebra program to calculate these constants.

2.1.2 Case study II: Cantilever beam

For a cantilever beam subjected to an tip-point load as shown in Figure 2:, the governing equation and boundary conditions are as below:

$$\frac{\partial^2}{\partial x^2}(-EI\frac{\partial^2 w}{\partial x^2}) = 0$$

$$w(0) = \bar{V}(L) - P = 0 \quad , \quad \bar{V} = EI\frac{\partial^3 w}{\partial x^3} \tag{11}$$

$$\frac{\partial w}{\partial x}(0) = \bar{M}(L) = 0$$

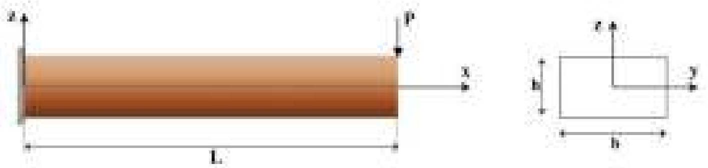

Figure 2: The cantilever nano-beam subjected to a tip-point load.

The solution for governing equations is as follows:

$$w = \frac{1}{6}C1x^3 + \frac{1}{2}C2x^2 + C3x + C4 \tag{12}$$

2.1.3 Shortcoming of formulation I

The normalized displacement (the ratio of the displacement to thickness) for both aforementioned case studies has been shown in Figure 3: and Figure 4:. These Figures show that the nonlocal nano-beam model of Reddy (2007) (formulation *I*) cannot capture the size effect significantly and its response is similar to that of the classical solution for the cantilever case and displays a small difference with the classical solution for the simply-supported case.

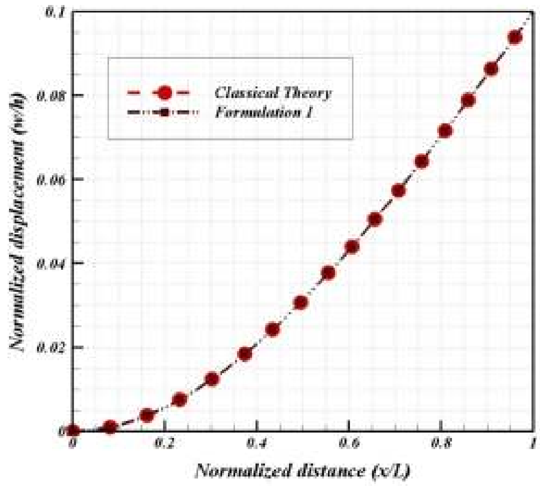

Figure 3: Normalized displacement of a cantilever beam according to the classical theory and formulation *I* with L=20h, b=2h, h=g and g=0.02 nm.

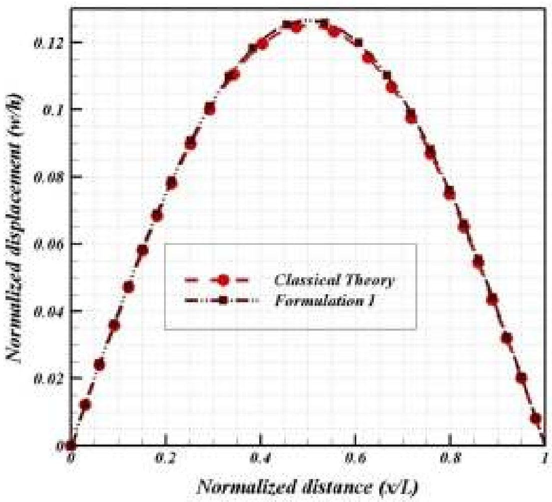

Figure 4: Normalized displacement of a simply-supported beam according to the classical theory and formulation *I* with L=20h, b=2h, h=g and g=0.02 nm.

As it was mentioned in section 2.1, it seems that this result is due to replacing nonlocal effect by the inertia gradient term in governing equations and boundary conditions. This is the basic shortcoming of the nonlocal elasticity theory in bending analysis of a nano-beam which cannot capture the size effect properly.

2.2 Strain gradient formulations of nano-beams (Formulation II)

In this subsection, the governing equations and boundary conditions of Euler-Bernoulli nano-beams are presented according to relations based on the strain gradient elasticity theory. The strain gradient Euler-Bernoulli beam formulation has been derived by Lazopoulos and Lazopoulos (2010). Hence for the sake of simplicity, similar governing equation and boundary conditions are considered here. Before Lazopoulos and Lazopoulos (2010), governing equation for Euler-Bernoulli beam had been developed by Papargyri-Beskou et al. (2003). But, one higher stress component which was missed in their paper has been considered by Lazopoulos and Lazopoulos (2010). The governing equations and boundary conditions without considering surface effect based on Lazopoulos and Lazopoulos (2010) work are expressed as follows:

$$\begin{aligned}
&E(I+g^2A)\frac{\partial^4 w}{\partial x^4}-g^2EI\frac{\partial^6 w}{\partial x^6}+q(x)=0\\
&\bar{V}=E(I+g^2A)\frac{\partial^3 w}{\partial x^3}-g^2EI\frac{\partial^5 w}{\partial x^5}\qquad or\qquad \delta w=0\\
&\bar{M}=E(I+g^2A)\frac{\partial^2 w}{\partial x^2}-g^2EI\frac{\partial^4 w}{\partial x^4}\qquad or\qquad \delta(\frac{\partial w}{\partial x})=0\\
&\bar{m}=g^2EI\frac{\partial^3 w}{\partial x^3}\qquad or\qquad \delta(\frac{\partial^2 w}{\partial x^2})=0
\end{aligned}\tag{13}$$

where A , $\bar{V}$, $\bar{M}$ and $\bar{m}$ are the cross section area, shear force, bending moment and higher order bending moment, respectively.

2.2.1 Case study I: Simply supported beam

To obtain the bending response of a simply supported beam which has been subjected to a constant distributed load as shown in Figure 1:, the following equations and boundary conditions has been extracted:

$$\begin{aligned}
&E(I+g^2A)\frac{\partial^4 w}{\partial x^4}-g^2EI\frac{\partial^6 w}{\partial x^6}+q=0\\
&w(0)=w(L)=0\\
&\frac{\partial w}{\partial x}(0)=\bar{M}(L)=0\\
&\bar{m}(0)=\frac{\partial^2 w}{\partial x^2}(L)=0
\end{aligned}\tag{14}$$

Solving the governing equations yields:

$$w=\frac{C1g^4I^2\mathrm{e}^{-\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{\left(Ag^2+I\right)^2}+\frac{C2g^4I^2\mathrm{e}^{\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{\left(Ag^2+I\right)^2}+\frac{1}{6}C3x^3+\frac{1}{2}C4x^2+C5x+C6+\frac{1}{24}\frac{qx^4}{E\left(Ag^2+I\right)} \qquad (15)$$

2.2.2 Case study II: Cantilever beam

The governing equations and boundary conditions for a cantilever beam subjected to a tip-point load as shown in Figure 2: are as follows:

$$\begin{aligned}
&E(I+g^2A)\frac{\partial^4 w}{\partial x^4}-g^2EI\frac{\partial^6 w}{\partial x^6}=0\\
&w(0)=\bar{V}(L)-P=0\\
&\frac{\partial w}{\partial x}(0)=\bar{M}(L)=0\\
&\frac{\partial^2 w}{\partial x^2}(0)=\frac{\partial^2 w}{\partial x^2}(L)=0
\end{aligned} \qquad (16)$$

The solution of governing equations is as below:

$$w=C1\mathrm{e}^{-\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}+C2\mathrm{e}^{\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}+C3x^3+C4x^2+C5x+C6 \qquad (17)$$

2.2.3 Shortcoming of formulation II

It should be noted that, although the strain gradient elasticity theory can be employed to study the static response of nano-beams, but like every boundary value problem, two sets of different boundary conditions can lead to two different static responses. The results obtained by methods of Lazopoulos and Lazopoulos (2010) and Papargyri-Beskou et al. (2003) show that although the supporting shape and loading conditions at both ends of the simply-supported beam are symmetric; but the predicted displacement curves (see Figure 5:) by their methods (Lazopoulos and Lazopoulos, 2010) and (Papargyri-Beskou et al., 2003) are not symmetric.

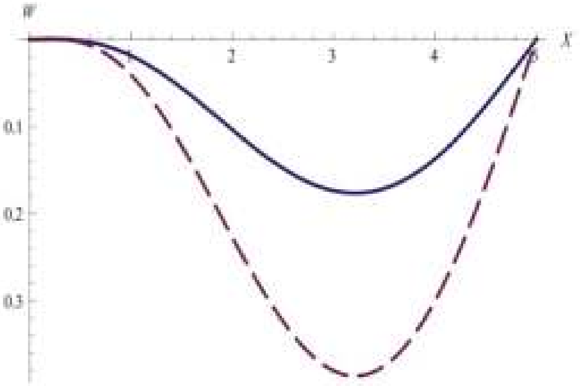

Figure 5: The elastic simply-supported micro-beam curve based on Lazopoulos and Lazopoulos (Lazopoulos and Lazopoulos, 2010) (solid line) and Papargyri-Beskou et al. (Papargyri-Beskou et al., 2003) (dashed line).

Also, the normalized displacement (the ratio of the displacement to thickness) for the case study I (simply-supported beam) has been shown in Figure 6:. The Figure shows that strain gradient nano-beam model of Lazopoulos and Lazopoulos (2010) (formulation *II*) does not converge to the classical elasticity solution in sufficiently large dimensions.

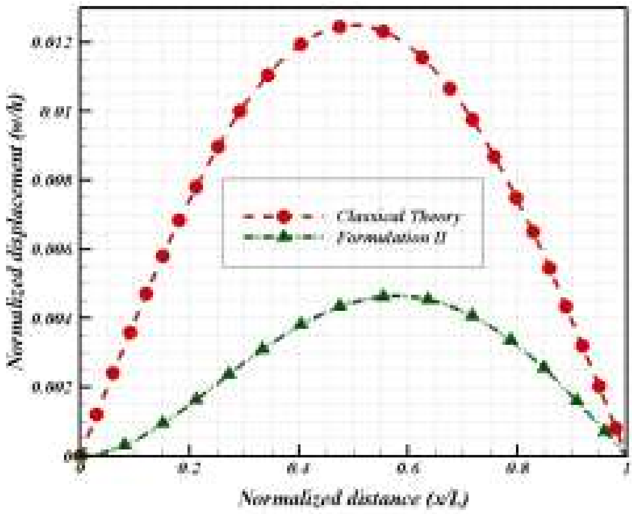

Figure 6: Normalized displacement of a simply-supported beam according to the classical theory and formulation *II* with L=20h, b=2h, h=10g and g=0.02 nm.

However, there is the basic question about boundary conditions in the strain gradient elasticity. How can we be confident that the new added boundary conditions based on the strain gradient elasticity theory (i.e., non-classical boundary conditions) have been chosen correctly to predict the real static response of the nano-beam? To cover all possible responses from the nano-scale to macro-scale, the boundary conditions should be chosen based on dimensional analysis. Generally speaking, the boundary conditions should be updated in every scale automatically. The present study follows this aim.

2.3 Strain gradient formulations of nano-beams with modified boundary conditions (Formulation III) - present work

In this subsection, the general form of the governing equation and boundary conditions of nano-beams are the same as Eq. (13). However, as discussed in section 2.2, in a paper by Lazopoulos and Lazopoulos (2010) for the case of a simply-supported beam, there is a contradiction about boundary conditions which deviates physical interpretation. To overcome this, the dimensional analysis has been employed and the boundary conditions have been modified based on this approach (see the Appendix).

2.3.1 Case study I: Simply supported beam

In the paper by Lazopolous and Lazopoulos (2010), one of the classical boundary conditions for a simply supported beam has been considered as follows:

$$\frac{\partial w}{\partial x}(0) = 0 \tag{18}$$

This condition considers a nonzero value for bending moment at x=0 (i.e. $M(0) \neq 0$). For a macro-beam, it can be clarified by employing the dimensional analysis (see the Appendix) that non-classical terms will be negligible and the equation of bending moment is the same as obtained by the classical elasticity theory:

$$M = EI\frac{\partial^2 w}{\partial x^2} \tag{19}$$

On the other hand, for non-classical boundary conditions, based on their method (Lazopoulos and Lazopoulos, 2010) the second derivative of the displacement does not vanish (i.e., $\frac{\partial^2 w}{\partial x^2}(0) \neq 0$). Hence, the bending moment at supported ends of the simply supported beam has a nonzero value either in small or large scales. This result is not compatible with the classical simply supported beam solution (i.e., large scale beam) which confirms the zero bending moment at supported ends.

So, to overcome the aforementioned incompatibility, the governing equations and meaningful boundary conditions which are compatible with the physical concepts in small or large scales should be as follows:

$$E(I+g^2A)\frac{\partial^4 w}{\partial x^4}-g^2EI\frac{\partial^6 w}{\partial x^6}+q=0$$

$$\begin{aligned}&w(0)=w(L)=0 \\ & \qquad\qquad\qquad\qquad \textit{classical boundary conditions} \\ &M(0)=M(L)=0 \\ &\frac{\partial^2 w}{\partial x^2}(0)=\frac{\partial^2 w}{\partial x^2}(L)=0 \qquad \textit{nonclassical boundary conditions}\end{aligned} \tag{20}$$

Then, the general solution is:

$$w=\frac{C1g^4I^2\mathrm{e}^{-\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{\left(Ag^2+I\right)^2}+\frac{C2g^4I^2\mathrm{e}^{\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{\left(Ag^2+I\right)^2}+\frac{1}{6}C3x^3+\frac{1}{2}C4x^2+C5x+C6+\frac{1}{24}\frac{qx^4}{E\left(Ag^2+I\right)} \tag{21}$$

For a macro-beam, governing equations and bending moment are converted to the classical one as:

$$\begin{aligned}&EI\frac{\partial^4 w}{\partial x^4}+q=0 \\ &M=EI\frac{\partial^2 w}{\partial x^2}\end{aligned} \tag{22}$$

Moreover, non-classical boundary conditions reduce to the classical one, and hence the following boundary conditions for a simply-supported beam subjected to a constant distributed load can be resulted as:

$$\begin{aligned}&w(0)=w(L)=0 \\ &M(0)=M(L)=0\end{aligned} \tag{23}$$

So, if the boundary conditions as mentioned in Eq. (20) are chosen, then it can be guaranteed that the classical solution will be obtained for a macro-beam.

2.3.2 Case study II: Cantilever beam

For a cantilever beam, one of the non-classical boundary conditions is as follows:

$$\frac{\partial^2 w}{\partial x^2}(0)=0 \tag{24}$$

The bending moment and the classical governing equation for this case are similar to Eq. (22) by equalizing *q(x)* to zero. The boundary condition as mentioned in Eq. (24) may leads to vanish the bending moment for the clamped end at the macro-scale (i.e., $M(0)=0$) and this is not in agreement with the classical cantilever beam boundary condition (i.e., $M(0)\neq 0$). So, instead of choosing the essential boundary condition at the clamped end, the natural boundary condition should be employed (i.e., $\bar{m}(0)=0$). Thus for a cantilever beam with a tip point load, the governing equation and boundary conditions are as follows:

$$\begin{aligned} & E(I+g^2A)\frac{\partial^4 w}{\partial x^4}-g^2EI\frac{\partial^6 w}{\partial x^6}=0 \\ & w(0)=\bar{V}(L)-P=0 \\ & \frac{\partial w}{\partial x}(0)=\bar{M}(L)=0 \\ & \bar{m}(0)=\frac{\partial^2 w}{\partial x^2}(L)=0 \end{aligned} \tag{25}$$

The general solution is as follows:

$$w=C1\mathrm{e}^{-\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}+C2\mathrm{e}^{\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}+C3x^3+C4x^2+C5x+C6 \tag{26}$$

Similar to the simply supported beam, the classical solution can be obtained at the macro-scale by using Eq. (25) for a cantilever beam.

2.4 Size-dependent elastic modulus of CNT based on strain gradient elasticity prediction

In this subsection, we use the solution of simply-supported beam based on formulations *II* and III to investigate the size-dependent elastic modulus of the CNT. Indeed, the beam model is employed to simulate the mechanical behavior of the CNT.

The strain components based on the Euler-Bernoulli theory are as follows:

$$\begin{aligned} & \varepsilon_{11}=\varepsilon_{xx}=-z\frac{d^2w}{dx^2} \\ & other\ \varepsilon_{ij}=0 \end{aligned} \tag{27}$$

$$\varepsilon_{xx} = -z\left(\frac{C1g^2 I\mathrm{e}^{-\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{Ag^2+I} + \frac{C2g^2 I\mathrm{e}^{\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{Ag^2+I} + C3x + C4 + \frac{1}{2}\frac{qx^2}{E\left(Ag^2+I\right)}\right) \tag{28}$$

The constitutive equation in a one dimensional case based on strain gradient elasticity is as follows:

$$\sigma_{xx} = E(\varepsilon_{xx} + g^2\frac{d^2\varepsilon_{xx}}{dx^2}) \tag{29}$$

$$\begin{aligned}\sigma_{xx} = &-z\left(\frac{C1g^2 I\mathrm{e}^{-\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{Ag^2+I} + \frac{C2g^2 I\mathrm{e}^{\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}}}{Ag^2+I} + C3x + C4 + \frac{1}{2}\frac{qx^2}{E\left(Ag^2+I\right)}\right)\\ &- g^2 z\left(C1\mathrm{e}^{-\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}} + C2\mathrm{e}^{\frac{\sqrt{Ag^2+I}x}{g\sqrt{I}}} + \frac{q}{E\left(Ag^2+I\right)}\right)\end{aligned} \tag{30}$$

Since the state of stress in the Euler-Bernoulli beam is one dimensional, so we can use Eq. (29) to obtain the effective stress. By employing Eqs. (14), (27) and (29) for formulation *II* and Eqs. (20), (27) and (29) for formulation *III*, the transverse displacement, strain component and effective stress are obtained. Then, the average stress and strain will be defined as follows:

$$\bar{\sigma}_{xx} = \frac{1}{L}\int_0^L \sigma_{xx}\, dx \quad , \quad \bar{\varepsilon}_{xx} = \frac{1}{L}\int_0^L \varepsilon_{xx}\, dx \tag{31}$$

Inspired the classical Hooke's law, the effective average Young's modulus of the CNT is defined as follows:

$$E^{nc} = \frac{\bar{\sigma}_{xx}}{\bar{\varepsilon}_{xx}} \tag{32}$$

The aforementioned equation was proposed by the authors to present the effect of material's micro-structure on Young's module overally. However, similar equations can be found in literature (Aifantis, 2008; Lim, 2010).

Eq. (32) is used for both formulations *II* and *III* to investigate the effect of boundary conditions in Young's modulus prediction based on the strain gradient elasticity theory. The Young's modulus relation based on formulations *II* and *III* are so long and the Maple computerized program was employed to obtain the explicit form and hence it is not reported here.

3 RESULTS AND DISCUSSION

In this section, a comparison between the bending responses of nano-beams based on three formulations will be accomplished and the appropriate form of the gradient elasticity and effect of proper boundary conditions will be clarified.

In Figure 7:, the normalized displacement (the ratio of the displacement to thickness) for a cantilever beam has been shown based on several formulations. The only difference between the curves based on formulations *II* and *III* is related to boundary conditions. The Figure shows that nonlocal nano-beam model of Reddy (2007) (formulation *I*) cannot capture the size effect significantly and its response is similar to that of the classical solution. But, the results obtained by the present formulation and Ref. (Lazopoulos and Lazopoulos, 2010) indicate a significant difference with the results of the classical cantilever beam and it verifies the importance of the size effect at the nano-scale which has been captured by the strain gradient elasticity theory. Generally speaking, since the structural stiffness has a direct relation to displacement, the structural stiffness based on strain gradient elasticity theory is higher than that of predicted by other methods such as classical and nonlocal elasticity theories. It means the beam is so stiff in the nano-scale that the classical and nonlocal elasticity theories are not able to model it.

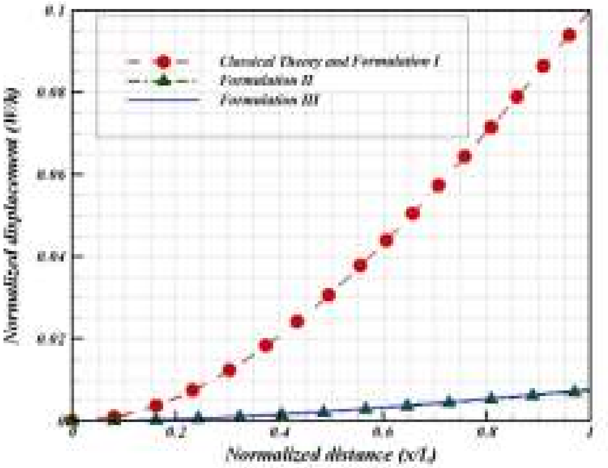

Figure 7: Normalized displacement of a cantilever beam according to different theories with L=20h, b=2h, h=g and g=0.02 nm.

Also, for a simply-supported beam subjected to a constant distributed load, normalized displacement has been shown based on different formulations in Figure 8:. In this case, formulation *I* displays a small difference with the classical one, while the other theories show sensible size effects. Again, it can be seen that strain gradient elasticity theory shows a significant increased stiffness in comparison with the classical and nonlocal elasticity theories. Also, based on the nonlocal elasticity prediction, the structural stiffness has been decreased in the nano-scale.

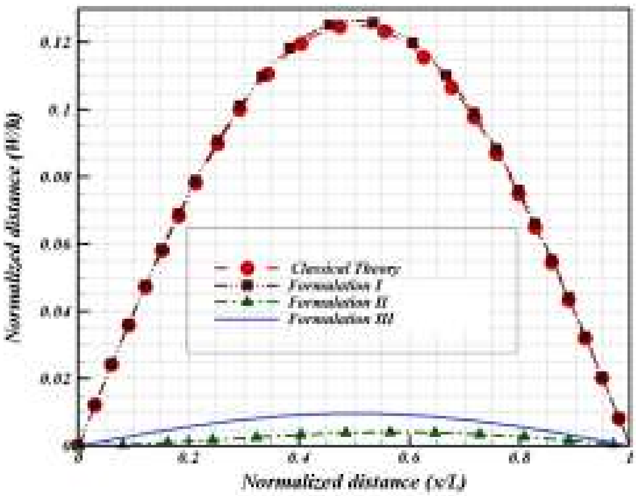

Figure 8: Normalized displacement of a simply-supported beam according to different theories with L=20h, b=2h, h=g and g=0.02 nm.

On the other hand, the size sensitivity of the aforementioned gradient elasticity theories can be seen in Figure 9:. As the beam thickness is varied, the present formulation shows more sensitivity in comparison with the nonlocal elasticity theory (formulation *I*) and its prediction coincides with the classical elasticity theory. Therefore, when dimensions of the beam lie on the macro-scale, the size effect is negligible and the classical elasticity theory is sufficient to predict the static behavior of beams. In addition, it can be seen that by increasing the thickness, the beam displacement is decreased.

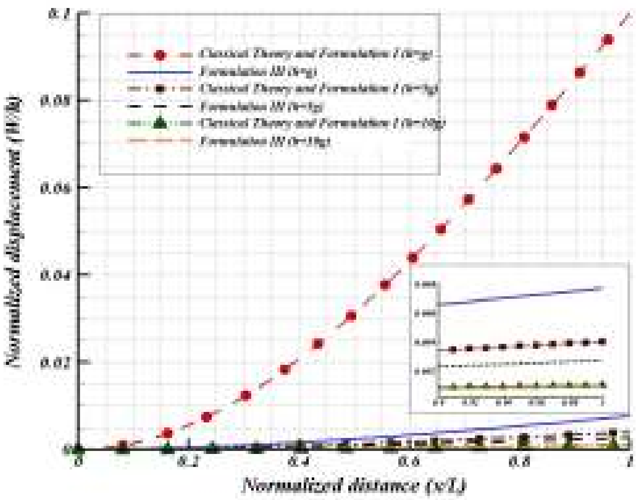

Figure 9: Normalized displacement of a cantilever beam according to different theories with L=20h, b=2h and g=0.02 nm.

Also, this sensitivity has been shown for a simply-supported beam in 0 and it is confirmed that with the variation of the beam thickness, the beam response is influenced by the size effect and for sufficiently large beams these effects are vanished. Also, it should be noted that the beam displacement decreases as the beam thickness increases.

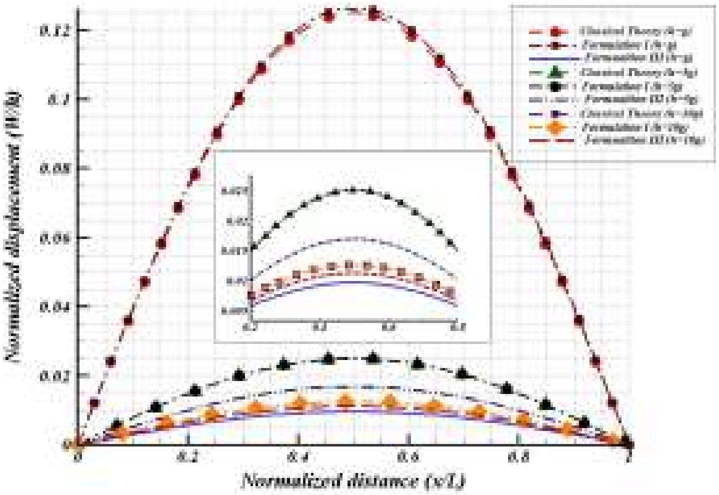

Figure 10: Normalized displacement of a simply-supported beam according to different theories with L=20h, b=2h and g=0.02 nm.

As it has been mentioned before, for sufficiently large beam the classical beam solution is enough and the non-classical solution is not needed. As it can be seen in Figure 10:, for a cantilever beam, both formulation *II* and formulation *III* reduce to the classical solution when the beam dimension is reached to the macro-scale. However, the result of the present formulation is more convergent to that of the classical method in comparison with other methods. This indicates the boundary conditions effect, which was considered by the present formulation very well. It must be noted that the boundary conditions effect is more effective for a simply-supported beam. As it has been shown in 0, as the beam thickness increases, the results of the present formulation are reduced to that of the classical beam solution. However, the other model does not predict the classical solution and the displacement of the nano-beam is still asymmetric. Therefore, it can be concluded that the effect of boundary conditions on the beam response is highly effective and it cannot be ignored. So, it must be concluded that not only choosing of the best non-classical elasticity theory has significant effects on prediction of static behavior of nano-beams, but also the boundary conditions' effects is so important.

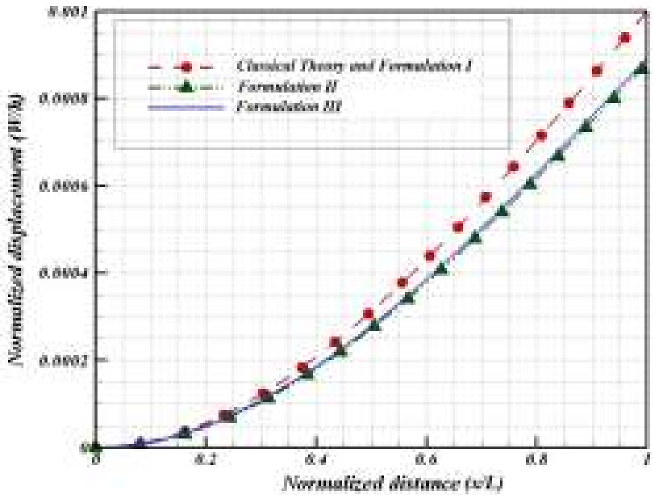

Figure 10:

Figure 11: Normalized displacement of a cantilever beam according to different theories with L=20h, b=2h, h=10g and g=0.02 nm.

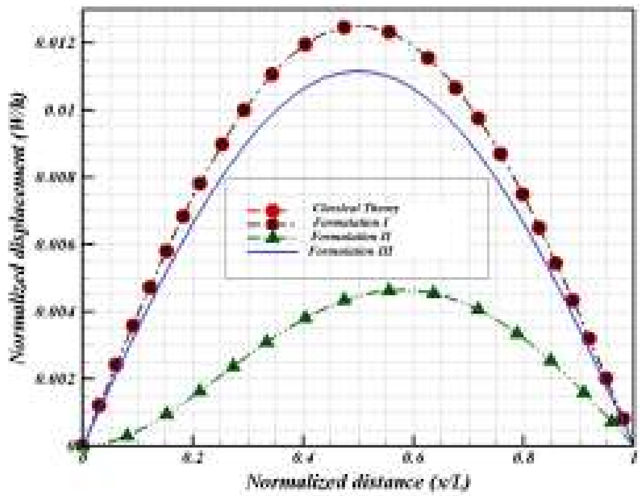

Figure 12: Normalized displacement of a simply-supported beam according to different theories with L=20h, b=2h, h=10g and g=0.02 nm.

To display the effect of boundary conditions on prediction of the Young's modulus of the CNT, based on strain gradient elasticity theory, the normalized Young modulus of the CNT (the ratio of the non-classical Young's modulus to the conventional Young's modulus) has been shown in 0. The radius of the CNT and its thickness are 0.678 nm and 0.34 nm, respectively.

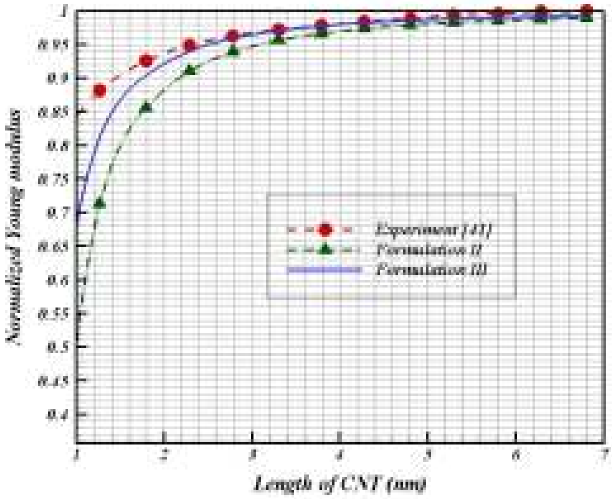

Figure 13: Normalized Young's modulus of CNT, based on strain gradient elasticity theories and experimental predictions.

As it can be found the Young's modulus predicted by formulation *III* is in a good agreement with experimental result (Sundararaghavan and Waas, 2011) in comparison with the prediction of formulation *II*.

4 CONCLUSIONS

In the present research a critique study on two gradient elasticity theories (i.e., nonlocal elasticity and strain gradient theories) for bending analysis of nano-beams was accomplished. As it was shown, the size effect had been replaced by inertia gradient in nonlocal elasticity theory (formulation *I*) and it leads to ignore the size effects on the bending solution of nano-beams, especially in a cantilever nano-beam. However, the two strain gradient elasticity based formulations (i.e., formulations *II* and *III*) demonstrate the size effects properly for bending analysis of nano-beams in comparison with the nonlocal elasticity theory (formulation *I*). In addition, due to choosing improper boundary conditions in strain gradient elasticity theory of nano-beams, some physical contradictions exist. For example, it is expected with increasing beam dimensions and disappearing size effects, the beam solution reduces to the classical solution. However, one of the strain gradient based formulation (i.e., formulation *II*) was not able to model this behavior properly for simply-supported nano-beams. Therefore, in the present research based on a dimensional analysis and clarifying size effects, the significance of the classical and non-classical terms in governing equations and boundary conditions were elucidated. As the results show, effect of boundary conditions in the determination of exact behavior of nano-beams is very significant and neglecting this effect results in a wrong bending response of nano-beams. In addition, as it was shown the effect of boundary conditions on prediction of the Young's modulus of nanostructures such as the CNT is crucial and the prediction of the strain gradient elasticity with modified boundary conditions is in a good agreement with experimental results.

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Appendix

In many papers (Ma et al., 2008, 2010; Wang et al., 2010), in both nonlocal or strain gradient elasticity models, it is stated that when the material length scale parameter vanishes, then governing equations reduce to the classical elasticity governing equations and hence classical beam solutions will be obtained. Generally speaking, in a sufficiently large dimension, the gradient or nonlocal effect will be disappeared and dimensional analysis can be employed to confirm this statement. In order to reach the classical solution for the static response of a large beam, not only governing equations but also the boundary conditions should be reduced to the classical form.

Firstly, one can consider a nano-beam which the orders of magnitude of its parameters are assumed as follows:

$$\begin{aligned} &O(w) = 10^{-3} n && O(I) = 10^{-12} n^4 \\ &O(x) = n && O(g^2) = n^2 \\ &O(A) = 10^{-6} n^2 && \end{aligned} \tag{33}$$

At very small scales, orders of magnitude for several terms in governing equations and boundary conditions can be computed as follows:

$$\begin{aligned} &O(g^2 A) = n^2 * 10^{-6} n^2 = 10^{-6} n^4 && O(g^2 I) = n^2 * 10^{-12} n^4 = 10^{-12} n^6 \\ &O(\frac{\partial^2 w}{\partial x^2}) = \frac{10^{-3} n}{n^2} = \frac{10^{-3}}{n} && O(\frac{\partial^3 w}{\partial x^3}) = \frac{10^{-3} n}{n^3} = \frac{10^{-3}}{n^2} \\ &O(\frac{\partial^4 w}{\partial x^4}) = \frac{10^{-3} n}{n^4} = \frac{10^{-3}}{n^3} && O(\frac{\partial^5 w}{\partial x^5}) = \frac{10^{-3} n}{n^5} = \frac{10^{-3}}{n^4} \\ &O(\frac{\partial^6 w}{\partial x^6}) = \frac{10^{-3} n}{n^6} = \frac{10^{-3}}{n^5} && \end{aligned} \tag{34}$$

Consequently the orders of magnitude for governing equation terms, shear force, bending moment and higher bending moment normalized by the Young's modulus are as follows:

$$\begin{aligned} &O((I + g^2 A)\frac{\partial^4 w}{\partial x^4} - g^2 I \frac{\partial^6 w}{\partial x^6}) = 10^{-15} n + 10^{-9} n + 10^{-15} n \\ &O(\frac{V}{E}) = 10^{-15} n^2 + 10^{-9} n^2 + 10^{-15} n^2 \\ &O(\frac{M}{E}) = 10^{-15} n^3 + 10^{-9} n^3 + 10^{-15} n^3 \\ &O(\frac{m}{E}) = 10^{-15} n^4 \end{aligned} \tag{3}$$

As it can be seen the $g^2 A$ term is very significant in comparison with other terms at micro-scale and this agrees with the conclusion made by Lazopoulos and Lazopoulos (Lazopoulos and Lazopoulos, 2010). Now consider a macro-beam and the orders of magnitude of its parameters are as follows:

$$\begin{aligned} &O(w) = 10^{-3} m && O(I) = 10^{-12} m^4 \\ &O(x) = m && O(g^2) = 10^{-12} m^2 \\ &O(A) = 10^{-6} m^2 && \end{aligned} \tag{35}$$

Again, orders of magnitude for several terms can be computed as below:

$$\begin{aligned}
&O(g^2 A) = 10^{-12} m^2 * 10^{-6} m^2 = 10^{-18} m^4 && O(g^2 I) = 10^{-12} m^2 * 10^{-12} m^4 = 10^{-24} m^6 \\
&O(\frac{\partial^2 w}{\partial x^2}) = \frac{10^{-3} m}{m^2} = \frac{10^{-3}}{m} && O(\frac{\partial^3 w}{\partial x^3}) = \frac{10^{-3} m}{m^3} = \frac{10^{-3}}{m^2} \\
&O(\frac{\partial^4 w}{\partial x^4}) = \frac{10^{-3} m}{m^4} = \frac{10^{-3}}{m^3} && O(\frac{\partial^5 w}{\partial x^5}) = \frac{10^{-3} m}{m^5} = \frac{10^{-3}}{m^4} \\
&O(\frac{\partial^6 w}{\partial x^6}) = \frac{10^{-3} m}{m^6} = \frac{10^{-3}}{m^5}
\end{aligned} \tag{36}$$

Finally, orders of magnitude for governing equation terms, shear force, bending moment and higher bending moment normalized by the Young's modulus of a macro-beam are as follows:

$$\begin{aligned}
&O((I + g^2 A)\frac{\partial^4 w}{\partial x^4} - g^2 I \frac{\partial^6 w}{\partial x^6}) = 10^{-15} m + 10^{-21} m + 10^{-27} m \\
&O(\frac{V}{E}) = 10^{-15} m^2 + 10^{-21} m^2 + 10^{-27} m^2 \\
&O(\frac{M}{E}) = 10^{-15} m^3 + 10^{-21} m^3 + 10^{-27} m^3 \\
&O(\frac{m}{E}) = 10^{-27} m^4
\end{aligned} \tag{37}$$

As it can be seen, $g^2 A$ and $g^2 I$ terms become insignificant and the classical terms are dominant in this case.