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Transverse Motions of Rectangular Plates Resting on Elastic Foundation and Under Concentrated Masses Moving at Varying Velocities

Abstract

This study concerns the dynamic characteristics of a prestressed isotropic, rectangular plate continuously supported by an elastic foundation and carrying accelerating mass M. Closed form solutions of the governing fourth order partial differential equations with variable and singular coefficients are presented. For the two-dimensional plate problem, the solution techniques is based on the double Fourier Finite Sine integral transformation, the expansion of the Dirac Delta function in series form, a modification of Struble's asymptotic method and the use of Fresnel sine and Fresnel cosine integrals. Numerical analyses in plotted curves are presented. The analyses reveal interesting results on the effect of structural parameters such as foundation moduli, rotatory inertia correction factor and prestressing forces on the dynamic behaviour of isotropic rectangular plate under the actions of concentrated masses moving at variable velocity. In particular it is found that the critical velocity of the travelling load which brings about the occurrence of a resonance state increases as the values of these structural parameters increase.

Keywords

Pretress, Isotropic, rectangular plate, concentrated masses, resonance, critical velocity.

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1 INTRODUCTION

The study of plate flexure under moving loads forms a very important structural element in Engineering design and construction. It has also become the objective of various investigations in the field of applied Mathematics and Physics. In general, problems of this type are mathematically complex when the inertial effect of the moving load is taken into consideration [Fryba, L., (1972), Milormir et al (1969), Sadiku et al, (1981), Gbadeyan and Oni (1995) Kargarmovin and Younesian (2004), Shahin and Mbakisya (2010), Awodola and Oni (2013), Awodola and Oni (2011), Omolofe (2013)].

In recent years, the speed and weight of commercial vehicles have been increased significantly. However, due to economical requirements, the bridge structures carrying these vehicles are fa bricated much lighter. These structures are therefore subjected to severe vibrations and dynamic stresses which are more than the corresponding static stresses. This has necessitated the quest for accurate evaluation of the vehicle-track interaction during the passage of the heavy subsystems. Nevertheless, it appears that most of the studies [Ismail (2011), Gbadeyan and Dada (2006), Wu et al (1987), Hsu (2009), Akin and Mofid (1989), Ali et al (2013), Shahed and Mohammad (2012), Sang-Jin (2013)] in this field focus on numerical simulations. Emphatically speaking, few studies concentrate on analytical developments. When these are available, the inertia effects of the heavy mass are neglected. It is well known that in a dynamical system as this, analytical method is desirable as solutions so obtained often shed light on vital information about the vibrating system [Fryba, L., (1972), Omolofe (2013),Stanisic et al (1974), Stanisic (1968), Hossein et al (2013), Atteshamuddin (2013), Gbadeyan and Oni (1992)].

Similarly, while much work are available in open literature on beam-type structure under moving masses, the vibration of plates under the actions of moving masses has so far received but scant attention. The first major breakthrough in this field of research was the work of Stanisic et al [Stanisic (1968] who solved the problem of a simply supported non-Mindlin plate under a multi-masses moving system by making use of an approximation of the Dirac delta function. Only the inertia terms that measures the effect of local acceleration in the direction of the deflection was considered. The method of solution was based on the Fourier Sine transform technique. The solutions so obtained were shown to converge very rapidly. The work of Stanisic et al was taken up much later by Gbadeyan and Oni [Gbadeyan and Oni (1992)] who investigated the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation and traversed by an arbitrary number of concentrated masses. All the components of the inertial terms were considered and the rectangular plate was taken to be simply supported. The deflection of the plates was calculated for several values of the foundation moduli and shown graphically as a function of time. More recently, study on an exact series solution for the transverse vibrations of rectangular plates with elastic boundary supports was carried out by [Li et al (2009)]. In this study, an analytical method was developed for the vibration analysis of rectangular plates with elastically restrained edges. Several numerical examples were presented to illustrate the excellent accuracy of their solution. Worthy of note, also, is the work of [Shadnam et al (2001)] who investigated the dynamics of plates under the influence of relatively large masses, moving along an arbitrary trajectory on the plate surface. As an example, the dynamic response of a rectangular plate, simply supported on all its edges, under a mass moving parallel to one of its sides and also travelling along circular trajectory is presented by means of operational calculus. Analysis showed that the response of structures due to moving mass, which has often been neglected in the past, must be properly taken into account because it often differs significantly from the moving force model.

It is remarked at this juncture that in all the aforementioned papers, the speeds at which these masses travel have been idealized to be uniform whereas, for practical purposes, these are not so. Such practical problems as acceleration and braking of automobile on roadways and highways bridges, taking off and landing of aircrafts on runway and breaking and acceleration forces in the calculation of rails and railway bridges in which the motion is not uniform, but a function of time have intensified the need for the study of the behaviour of structures under the action of loads moving at varying velocities. This class of problems was first tackled by [Lowan (1935)] who solved the problem of the transverse oscillations of beams under the actions of moving variable loads. Much later, are the studies by [Kokhmanyuk and Filippov (1967)], [Huang and Thambiratnam (2001)], [Oni (2004)] and [Oni and Omolofe (2011)].

In our recent paper [Oni and Omolofe (2011)], effort was made to investigate the dynamic response of prestressed Rayleigh beam resting on Elastic foundation and subjected to masses travelling at varying velocities. The objective of this paper is to extend this work to the dynamic behaviour of plate-type structures and as in the previous paper obtain analytical solutions. This paper therefore, investigates the transverse motions of rectangular plate resting on elastic foundation and under the actions of concentrated masses moving at varying velocities.

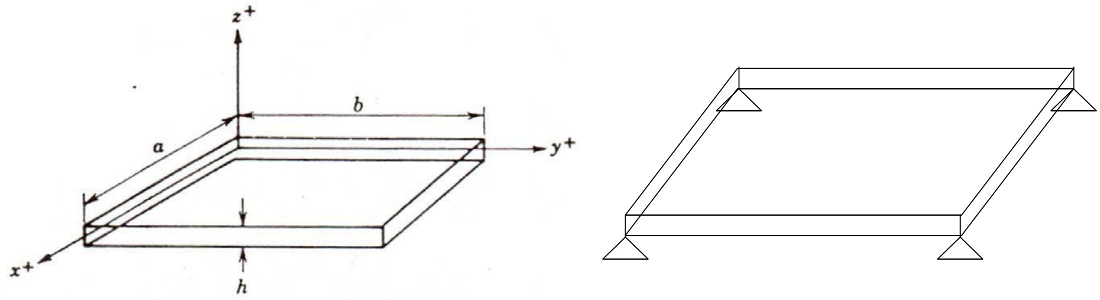

Figure 1: Schematic diagram of a rectangular plate on moving load.

2 FORMULATION OF THE GOVERNING EQUATION

The problem of the flexural motion of isotropic, structurally prestressed rectangular plate resting on an elastic foundation and carrying a mass M is investigated. A rectangular plate of thickness h and lateral dimensions L_x and L_y (respectively in the x and y direction in the rectangular axis) under the actions of load $P(x,y,t)$ of mass M traveling from point $y = y_0$ on the plate along a straight line parallel to the x-axis with a non-uniform velocity as shown in fig1* is considered. Neglecting damping and the effects of shear deformation, according to the classical two-dimensional theory of flexural motions of isotropic elastic rectangular plate, the transverse displacement $U(x,y,t)$, of the mid-surface of the rectangular plate exhibiting anisotropic prestress when the inertia effect of the accelerating mass on the transverse response of the rectangular plate is taken into consideration is governed by the fourth order partial differential equation given by

$$\left(D\nabla^2 - \mu R_o \frac{\partial^2}{\partial t^2}\right)\nabla^2 U(x,y,t) - \left(N_x \frac{\partial^2 U(x,y,t)}{\partial x^2} + N_y \frac{\partial^2 U(x,y,t)}{\partial y^2}\right) + \mu \frac{\partial^2 U(x,y,t)}{\partial t^2} + KU(x,y,t) = Mg\delta[x - x(t)]\delta(y - y_0)\left[1 - \frac{\Delta^*}{g}U(x,y,t)\right] \tag{1}$$

where $D = \frac{Eh^3}{12(1-\nu)}$ is the bending rigidity of the plate, E is the young modulus, ν is the poisson's ratio $(\nu < 1)$, μ is the mass of the plate per unit length, x is the position coordinate in x-direction , y is the position coordinate in y-direction, t is the time and R_o is the measure of rotatory inertia, Nx and Ny are respectively the prestressing forces in x and y directions respectively and where the Δ^* is the convective acceleration operator defined as

$$\Delta^* = \frac{\partial^2}{\partial x^2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{\partial^2}{\partial y^2}\left(\frac{dy(t)}{dt}\right)^2 + \frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial x \partial y}\frac{dx(t)}{dt}\frac{dy(t)}{dt} + 2\frac{\partial^2}{\partial x \partial t}\frac{dx(t)}{dt} + 2\frac{\partial^2}{\partial y \partial t}\frac{dy(t)}{dt} + \frac{\partial}{\partial x}\frac{d^2 x(t)}{dt^2} + \frac{\partial}{\partial y}\frac{d^2 y(t)}{dt^2} \tag{2}$$

Using equations (2) in equation (1), one obtains

$$D\left[\frac{\partial^4 U(x,y,t)}{\partial x^4} + 2\frac{\partial^4 U(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 U(x,y,t)}{\partial y^4}\right] + \mu \frac{\partial^2 U(x,y,t)}{\partial t^2} - N_x \frac{\partial^2 U(x,y,t)}{\partial x^2} - N_y \frac{\partial^2 U(x,y,t)}{\partial y^2} - \mu R_o\left[\frac{\partial^4 U(x,y,t)}{\partial x^2 \partial t^2} + \frac{\partial^4 U(x,y,t)}{\partial y^2 \partial t^2}\right] + KU(x,y,t) = P_F(x,y,t)\delta[x - x(t)]\delta[y - y(t)]\left[1 - \frac{1}{g}\left\{\frac{\partial^2 U(x,y,t)}{\partial x^2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{\partial^2 U(x,y,t)}{\partial y^2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{\partial^2 U(x,y,t)}{\partial t^2} + 2\frac{\partial^2 U(x,y,t)}{\partial x \partial y}\frac{dx(t)}{dt}\frac{dy(t)}{dt} + 2\frac{\partial^2 U(x,y,t)}{\partial x \partial t}\frac{dx(t)}{dt} + 2\frac{\partial^2 U(x,y,t)}{\partial y \partial t}\frac{dy(t)}{dt} + \frac{\partial U(x,y,t)}{\partial x}\frac{d^2 x(t)}{dt^2} + \frac{\partial U(x,y,t)}{\partial y}\frac{d^2 y(t)}{dt^2}\right\}\right. \tag{3}$$

The rectangular plate under consideration is simply supported and such, the deflection and the moments at the edges $x = 0$, $x = L_x$, $y = 0$ and $y = L_y$ vanish. Thus, the boundary conditions are

$$U(0,y,t) = 0; \quad U(L_x,y,t) = 0; \quad U(x,0,t) = 0; \quad U(x,L_y,t) = 0$$
$$\frac{\partial^2 U(0,y,t)}{\partial x^2} = 0; \quad \frac{\partial^2 U(L_x,y,t)}{\partial x^2} = 0; \quad \frac{\partial^2 U(x,0,t)}{\partial y^2} = 0; \quad \frac{\partial^2 U(x,L_y,t)}{\partial y^2} = 0 \tag{4}$$

and the initial conditions without any loss of generality are taken to be

$$U(x,y,t)\Big|_{t=0} = 0 = \frac{\partial U(x,y,t)}{\partial t}\Big|_{t=0} \tag{5}$$

where $\delta(\cdot)$ is the dirac delta function and

$$x(t) = x_0 + ct + \frac{1}{2}at^2 \ , \ y(t) = y_0 \tag{6}$$

where x_0 and y_0 are the initial positions in the x and y directions respectively, c is the initial velocity and a is the acceleration of the traveling load. Equation (9) expresses a uniformly accelerated (a >0) or uniformly decelerated (a < 0) motion. Time t is assumed to be limited to that interval of time within which the mass M is on the plate, that is

$$0 \le x(t) \le L \tag{7}$$

Thus, in view of equations (6) and (7), equation (3) can be written as

$$\begin{aligned} & D\left[\frac{\partial^4 U(x,y,t)}{\partial x^4} + 2\frac{\partial^4 U(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 U(x,y,t)}{\partial y^4}\right] + \mu \frac{\partial^2 U(x,y,t)}{\partial t^2} - N_x \frac{\partial^2 U(x,y,t)}{\partial x^2} - N_y \frac{\partial^2 U(x,y,t)}{\partial y^2} \\ & - \mu R_o \left[\frac{\partial^4 U(x,y,t)}{\partial x^2 \partial t^2} + \frac{\partial^4 U(x,y,t)}{\partial y^2 \partial t^2}\right] + KU(x,y,t) \\ & + M\delta\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\delta[y - y_0]\left[(c+at)^2 \frac{\partial^2 U(x,y,t)}{\partial x^2} + \frac{\partial^2 U(x,y,t)}{\partial t^2}\right. \\ & \left. + 2(c+at)\frac{\partial^2 U(x,y,t)}{\partial x \partial t} + a\frac{\partial U(x,y,t)}{\partial x}\right] = Mg\delta\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\delta[y - y_0] \end{aligned} \tag{8}$$

Equation (8) is the fourth order partial differential equation governing the flexural motion of the prestressed rectangular plate on Winkler foundation and under the actions of load moving at non-uniform velocity.

3 ANALYTICAL SOLUTION PROCEDURES

The double Fourier finite sine integral transformation with respect to the spatial coordinates x and y discussed in [Gbadeyan and Oni (1995)] would be employed to transform the governing partial differential equation to a second order differential equation. Subsequently, as in the previous paper [Oni and Omolofe (2011)], the asymptotic method of Struble's will be used to simplify this equation. The double Fourier finite sine integral transformation is defined by the following relations between the original functions $U(x,y,t)$ and its transform $\overline{U}(j,k,t)$, that is,

$$\overline{U}(j,k,t)=\int_0^{L_y}\int_0^{L_x} U(x,y,t)\sin\frac{j\pi x}{L_x}\sin\frac{j\pi y}{L_y}\,dxdy \tag{9}$$

with the inverse

$$U(x,y,t)=\frac{4}{L_xL_y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\overline{U}(j,k,t)\sin\frac{j\pi x}{L_x}\sin\frac{k\pi y}{L_y} \tag{10}$$

Using (9) and (10), and taking into account the boundary conditions (4) the governing equation (8) is transformed to take the form

$$\begin{aligned}
&\overline{U}_{tt}(j,k,t)+\frac{\Omega_{j,k}^2}{\mu}\overline{U}(j,k,t)+\frac{D_m}{\mu}Z(0,l,t)-\frac{N_x}{\mu}\sum_{p=1}^{\infty}\overline{U}(p,k,t)H_a^*(p,j)+\frac{N_y}{\mu}\frac{k^2\pi^2}{L_y^2}\sum_{p=1}^{\infty}\overline{U}(p,k,t)H_b^*(p,j)\\
&-R_0\sum_{p=1}^{\infty}\overline{U}_{tt}(p,k,t)H_a^*(p,j)+R_o\frac{k^2\pi^2}{L_y^2}\sum_{p=1}^{\infty}\overline{U}_{tt}(p,k,t)H_b^*(p,j)+\frac{K}{\mu}\overline{U}(j,k,t)+\Gamma_0^*\Bigg[(c+at)^2\Bigg\{\sum_{p=1}^{\infty}\overline{U}(p,k,t)H_a^*(p,j)\\
&+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\overline{U}(p,q,t)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_a^*(p,j)+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}(p,k,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)H_c^*(p,j,n)\\
&+4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}(p,q,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_c^*(p,j,n)\Bigg\}\\
&+2(c+at)\Bigg\{\sum_{p=1}^{\infty}\overline{U}_t(p,k,t)H_e^*(p,j)+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\overline{U}_t(p,q,t)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_e^*(p,j)\\
&+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}_t(p,k,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)H_f^*(p,j,n)\\
&+4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}_t(p,q,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_f^*(p,j,n)\Bigg\}\\
&+\sum_{p=1}^{\infty}\overline{U}_{tt}(p,k,t)H_b^*(p,j)+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\overline{U}_{tt}(p,q,t)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_b^*(p,j)\\
&+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}_{tt}(p,k,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)H_d^*(p,j,n)\\
&+4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}_{tt}(p,q,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_d^*(p,j,n)\\
&+a\sum_{p=1}^{\infty}\overline{U}(p,k,t)H_e^*(p,j)+2a\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\overline{U}(p,q,t)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_e^*(p,j)\\
&+2a\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}(p,k,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)H_f^*(p,j,n)\\
&+4a\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\overline{U}(p,q,t)\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}H_f^*(p,j,n)\Bigg]\\
&=\frac{Mg}{\mu}\sin\frac{k\pi y_0}{L_y}\sin\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)
\end{aligned} \tag{11}$$

where

$$H_a^*(p,j) = -\frac{p^2\pi^2}{L_x^3}\int_0^{L_x} \sin\frac{p\pi x}{L_x}\sin\frac{j\pi x}{L_x}dx,$$

$$H_b^*(p,j) = \frac{2}{L_x}\int_0^{L_x} \sin\frac{p\pi x}{L_x}\sin\frac{j\pi x}{L_x}dx,$$

$$H_c^*(p,j) = -\frac{2p^2\pi^2}{L_x^3}\int_0^{L_x} \cos\frac{n\pi x}{L_x}\sin\frac{p\pi x}{L_x}\sin\frac{j\pi x}{L_x}dx, \tag{12}$$

$$H_d^*(p,j) = \frac{2}{L_x}\int_0^{L_x} \cos\frac{n\pi x}{L_x}\sin\frac{p\pi x}{L_x}\sin\frac{j\pi x}{L_x}dx,$$

$$H_e^*(p,j) = \frac{2p\pi}{L_x^2}\int_0^{L_x} \cos\frac{p\pi x}{L_x}\sin\frac{j\pi x}{L_x}dx,$$

$$H_f^*(p,j) = \frac{2p\pi}{L_x^2}\int_0^{L_x} \cos\frac{n\pi x}{L_x}\cos\frac{p\pi x}{L_x}\sin\frac{j\pi x}{L_x}dx$$

Equation (11) is now the fundamental equation of our problem when the non-Mindlin's rectangular plate has simple supports at all its edges. In what follows, two special cases of the equation (11) above are considered namely the moving force and the moving mass problems.

a. The Moving Force Model

To obtain the moving force model of our dynamical system when the isotropic rectangular plate has simple support at all its edges, Γ_0^* is set to zero in equation (11) and this leads to

$$\begin{aligned}&\overline{U}_{tt}(j,k,t) + \frac{\Omega_{j,k}^2}{\mu}\overline{U}(j,k,t) + \frac{D_m}{\mu}Z\left(0,L_x,L_y,t\right) + \frac{N_x}{\mu}\frac{j^2\pi^2}{L_x^2}\overline{U}(j,k,t) + \frac{N_y}{\mu}\frac{k^2\pi^2}{L_y^2}\overline{U}(j,k,t)\\&+R_o\frac{j^2\pi^2}{L_y^2}\overline{U}_{tt}(j,k,t) + R_0\frac{k^2\pi^2}{L_y^2}\overline{U}_{tt}(j,k,t) + \frac{K}{\mu}\overline{U}(j,k,t) = \frac{Mg}{\mu}\sin\frac{k\pi y_0}{L_y}\sin\frac{j\pi}{L_x}\left(x_0 + ct + \frac{1}{2}at^2\right)\end{aligned} \tag{13}$$

This is an approximate model which assumes the inertia effect of the moving mass as negligible. It is straightforward to show that equation (13) after some simplifications and rearrangements yields

$$\overline{U}_{tt}(j,k,t) + \gamma_{JKF}^2\overline{U}(j,k,t) = P_{JK}^*\sin\frac{j\pi}{L_x}\left(x_0 + ct + \frac{1}{2}at^2\right) \tag{14}$$

where

$$\gamma_{JKF}^2 = \frac{\frac{\Omega_{JK}^2}{\mu} + \left(\frac{N_x}{\mu} \frac{j^2\pi^2}{L_x^2} + \frac{N_y}{\mu} \frac{k^2\pi^2}{L_y^2} \right) + \frac{K}{\mu}}{1 + R_0\left(\frac{j^2\pi^2}{L_x^2} + \frac{k^2\pi^2}{L_y^2} \right)} \quad \text{and} \quad P_{JK}^* = \frac{Mg \sin \frac{k\pi y_0}{L_y}}{1 + R_o\left(\frac{j^2\pi^2}{L_x^2} + \frac{k^2\pi^2}{L_y^2} \right)} \tag{15}$$

Equation (14) when solved in conjunction with the initial conditions, one obtains an expression for $\overline{U}_{tt}(j,k,t)$ which on inversion yields

$$\begin{aligned} U(x,y,t) = \frac{4}{L_x L_y} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{P_{JK}^* \sqrt{\pi}}{2\gamma_{JK} \sqrt{2a_0}} \Bigg(& Sin\gamma_{JKF} t \Bigg[Cos\left(\frac{b_2^2}{4a_0} - c_0 \right) S\left(\frac{b_2 + 2a_0 t}{\sqrt{2\pi a_0}} \right) - C\left(\frac{b_2 + 2a_0 t}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_2^2}{4a_0} - c_0 \right) \\ & + Cos\left(\frac{b_1^2}{4a_0} - c_0 \right) S\left(\frac{b_1 + 2a_0 t}{\sqrt{2\pi a_0}} \right) - C\left(\frac{b_1 + 2a_0 t}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_1^2}{4a_0} - c_0 \right) - Cos\left(\frac{b_2^2}{4a_0} - c_0 \right) S\left(\frac{b_2}{\sqrt{2\pi a_0}} \right) \\ & + C\left(\frac{b_2}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_2^2}{4a_0} - c_0 \right) - Cos\left(\frac{b_1^2}{4a_0} - c_0 \right) S\left(\frac{b_1}{\sqrt{2\pi a_0}} \right) + C\left(\frac{b_1}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_1^2}{4a_0} - c_0 \right) \Bigg] \\ & - Cos\gamma_{JKF} t \Bigg[Cos\left(\frac{b_1^2}{4a_0} - c_0 \right) C\left(\frac{b_1 + 2a_0 t}{\sqrt{2\pi a_0}} \right) + S\left(\frac{b_1 + 2a_0 t}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_1^2}{4a_0} - c_0 \right) - Cos\left(\frac{b_2^2}{4a_0} - c_0 \right) C\left(\frac{b_2 + 2a_0 t}{\sqrt{2\pi a_0}} \right) \\ & - S\left(\frac{b_2 + 2a_0 t}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_2^2}{4a_0} - c_0 \right) - Cos\left(\frac{b_1^2}{4a_0} - c_0 \right) C\left(\frac{b_1}{\sqrt{2\pi a_0}} \right) - S\left(\frac{b_1}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_1^2}{4a_0} - c_0 \right) \\ & + Cos\left(\frac{b_2^2}{4a_0} - c_0 \right) C\left(\frac{b_2}{\sqrt{2\pi a_0}} \right) + S\left(\frac{b_2}{\sqrt{2\pi a_0}} \right) Sin\left(\frac{b_2^2}{4a_0} - c_0 \right) \Bigg] \Bigg) \left(\sin \frac{k\pi y}{L_y} \sin \frac{j\pi x}{L_x} \right) \end{aligned} \tag{16}$$

which represents the transverse displacement response to concentrated forces moving at variable velocities of a simply-supported isotropic rectangular plate incorporating rotatory inertia correction factor.

The functions $S(x)$ and $C(x)$ are the Fresnel Sine and Cosine functions respectively. For real values of argument x, the values of the Fresnel integrals $S(x)$ and $C(x)$ are real.

b. The moving Mass Model

If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus, $\Gamma_0^* \neq 0$ and one is required to solve the entire equation (11) when no term of the coupled differential equation is neglected. This is termed the moving mass problem. Unlike in the case of the moving force, an exact analytical solution to this equation is not possible. Thus, one resorts to an approximate analytical technique due to Struble discussed in [29, 30]. By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the mass of the load. To this end, equation (11) is rearranged to take the form

$$\overline{U}_{tt}(j,k,t)+\frac{\Gamma_0^* G_2(j,k,t)}{1+\Gamma_0^* G_1(j,k,t)}\overline{U}_t(j,k,t)+\frac{\left(\gamma_{JKF}^2+\Gamma_0^* G_3(j,k,t)\right)}{1+\Gamma_0^* G_1(j,k,t)}\overline{U}(j,k,t)$$
$$+\frac{\Gamma_0^*}{1+\Gamma_0^* G_1(j,k,t)}\Bigg[(c+at)^2\Bigg\{-2\sum_{\substack{p=1\\p\neq j}}^{\infty}\frac{p^2\pi^2}{L_x^2}\sin\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{p\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\overline{U}(p,k,t)$$
$$-4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\frac{p^2\pi^2}{L_x^2}\sin\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{p\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}\overline{U}(p,k,t)\Bigg\}$$
$$+2(c+at)\Bigg\{\sum_{\substack{p=1\\p\neq j}}^{\infty}\frac{4jp}{L_x\left(j^2-p^2\right)}\overline{U}_t(p,k,t)+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\frac{4jp}{L_x\left(j^2-p^2\right)}\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}\overline{U}_t(p,k,t)$$
$$+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{nn=1}^{\infty}\frac{4jp\left(j^2-p^2-n^2\right)}{L_x\left[(j+p)^2-n^2\right]\left[(j-p)^2-n^2\right]}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\overline{U}_t(p,k,t)$$
$$+4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=1}^{\infty}\frac{4jp\left(j^2-p^2-n^2\right)}{L_x\left[(j+p)^2-n^2\right]\left[(j-p)^2-n^2\right]}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}\overline{U}_t(p,k,t)\Bigg\} \quad (17)$$
$$+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sin\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{p\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\overline{U}_{tt}(p,k,t)$$
$$+4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sin\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{p\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}\overline{U}_{tt}(p,k,t)$$
$$+a\sum_{\substack{p=1\\p\neq j}}^{\infty}\frac{4jp}{L_x\left(j^2-p^2\right)}\overline{U}(p,k,t)+2a\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\frac{4jp}{L_x\left(j^2-p^2\right)}\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}\overline{U}(p,k,t)$$
$$+2a\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{n=1}^{\infty}\frac{4jp\left(j^2-p^2-n^2\right)}{L_x\left[(j+p)^2-n^2\right]\left[(j-p)^2-n^2\right]}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\overline{U}(p,k,t)$$
$$+4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=1}^{\infty}\frac{4jp\left(j^2-p^2-n^2\right)}{L_x\left[(j+p)^2-n^2\right]\left[(j-p)^2-n^2\right]}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin\frac{k\pi y_0}{L_y}\sin\frac{q\pi y_0}{L_y}\overline{U}_t(p,k,t)$$
$$=\Gamma_0^* L_x L_y g\sin\frac{k\pi y_0}{L_y}\sin\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)$$

where

$$G_1(j,k,t)=1+2\sin^2\frac{k\pi y_0}{L_y}+\left[1-\cos\frac{2\pi j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]+2\left[1-\cos\frac{2\pi j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]\sin^2\frac{k\pi y_0}{L_y}$$

$$G_2(j,k,t)=2(c+at)\left\{\sum_{n=1}^{\infty}\frac{8j^2}{L_x\left(4j^2-n^2\right)}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)+\sum_{n=1}^{\infty}\frac{16j^2}{L_x\left(4j^2-n^2\right)}\sin^2\frac{k\pi y_0}{L_y}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right\}$$

and

$$G_3(j,k,t)=(c+at)^2\left\{-\frac{j^2\pi^2}{L_x^2}-\frac{j^2\pi^2}{L_x^2}\sin^2\frac{k\pi y_0}{L_y}-\frac{j^2\pi^2}{L_x^2}\left[1-\cos\frac{2\pi j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]\right.$$
$$-2\frac{j^2\pi^2}{L_x^2}\sin^2\frac{k\pi y_0}{L_y}\left[1-\cos\frac{2\pi j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]+\sum_{n=1}^{\infty}\frac{8j^2}{L_x(4j^2-n^2)}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right) \tag{18}$$
$$\left.+\sum_{n=1}^{\infty}\frac{16j^2}{L_x(4j^2-n^2)}\sin^2\frac{k\pi y_0}{L_y}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right\}$$

At this juncture, the homogeneous part of equation (17) is first considered and a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is sought. An equivalent free system operator defined by the modified frequency replaces equation (17). To this effect, one considers a parameter $\eta_1^* < 1$ for any arbitrary mass ratio Γ_0^* defined as

$$\eta_1^* = \frac{\Gamma_0^*}{1+\Gamma_0^*} \tag{19}$$

from which it is evident that

$$\Gamma_0^* = \eta_1^* + O\left(\eta_1^{*2}\right) \tag{20}$$

Noting that

$$\frac{1}{1+\Gamma_0^* G_1} = \left(1+\Gamma_0^* G_1\right)^{-1} = 1-\Gamma_0^* G_1 + \Gamma_0^{*2} G_1^2 - \Gamma_0^{*3} G_1^3 \pm \ldots\ldots\ldots\ldots, \tag{21}$$

whenever,

$$\left|\Gamma_0^* G_1\right| < 1., \tag{22}$$

When $\eta_1^* = 0$, a case corresponding to the case when the inertia effect of the mass of the system is neglected is obtained, then the solution of (17) can be written in the form

$$\overline{U}(j,k,t) = C_2 \cos\left(\gamma_{JKF} t - \phi_{jk}\right) \tag{23}$$

where C_2 and ϕ_{jk} are constants.

Since $\eta_1^* < 1$, Struble's technique requires that the solution of the homogeneous part of equation (17) be written in an asymptotic form, namely

$$\overline{U}(j,k,t) = A(j,k,t)\cos\left[\gamma_{JKF} t - \phi(j,k,t)\right] + \eta_1^* \overline{U}_1(j,k,t) + O\left(\eta_1^{*2}\right) \tag{24}$$

To obtain the modified frequency of our dynamical system, equation (24) and its first and second derivatives are substituted into the homogeneous part of equation (17). While taking into account (20) and (21) and retaining terms to $O(\eta_1^*)$ only.

Thus after some simplifications and rearrangements, one obtains

$$
\begin{aligned}
&-2\gamma_{JKF}\dot{A}(j,k,t)\sin[\gamma_{JKF}t-\phi(j,k,t)]+2\gamma_{JKF}A(j,k,t)\dot{\phi}(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&-2\gamma_{JKF}(c+at)\eta_1^*A(j,k,t)\left(\sum_{n=1}^{\infty}\frac{8j^2}{L_x(4j^2-n^2)}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right.\\
&\left.+\sum_{n=1}^{\infty}\frac{16j^2}{L_x(4j^2-n^2)}\sin^2\frac{k\pi y_0}{L_y}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right)\sin[\gamma_{JKF}t-\phi(j,k,t)]\\
&-\eta_1^*\gamma_{JKF}^2A(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]-2\eta_1^*\gamma_{JKF}^2A(j,k,t)\sin^2\frac{k\pi y_0}{L_y}\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&-\eta_1^*\gamma_{JKF}^2A(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]+\eta_1^*\gamma_{JKF}^2A(j,k,t)\cos\frac{2\pi j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&-2\eta_1^*\gamma_{JKF}^2A(j,k,t)\sin^2\frac{k\pi y_0}{L_y}\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&+2\eta_1^*\gamma_{JKF}^2A(j,k,t)\sin^2\frac{k\pi y_0}{L_y}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&-\eta_1^*\frac{j^2\pi^2}{L_x^2}(c+at)^2A(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&-2\eta_1^*\frac{j^2\pi^2}{L_x^2}\sin^2\frac{k\pi y_0}{L_y}(c+at)^2A(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&-\eta_1^*\frac{j^2\pi^2}{L_x^2}(c+at)^2A(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&+\eta_1^*\frac{j^2\pi^2}{L_x^2}(c+at)^2A(j,k,t)\cos\frac{2\pi j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&-2\eta_1^*\frac{j^2\pi^2}{L_x^2}(c+at)^2A(j,k,t)\sin^2\frac{k\pi y_0}{L_y}\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&+2\eta_1^*\frac{j^2\pi^2}{L_x^2}A(j,k,t)\sin^2\frac{k\pi y_0}{L_y}\cos\frac{2\pi j}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&+\eta_1^*A(j,k,t)\sum_{n=1}^{\infty}\frac{8aj^2}{L_x(4j^2-n^2)}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\cos[\gamma_{JKF}t-\phi(j,k,t)]\\
&+\eta_1^*A(j,k,t)\sum_{n=1}^{\infty}\frac{16aj^2}{L_x(4j^2-n^2)}\sin^2\frac{k\pi y_0}{L_y}\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\cos[\gamma_{JKF}t-\phi(j,k,t)]=0\,,
\end{aligned}
\tag{25}
$$

to $O(\eta_1^*)$ only.

To obtain the variational equations, one equates the coefficients of $\cos[\gamma_{JKF}t-\phi(j,k,t)]$ and $\sin[\gamma_{JKF}t-\phi(j,k,t)]$ on both sides of equation (25). To do this, the following trigonometric identities are taken into account

$$\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\sin[\gamma_{JKF}t-\phi(j,k,t)]=\frac{1}{2}\left\{\sin\left[\gamma_{JKF}t-\phi(j,k,t)+\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]\right.$$
$$\left.-\sin\left[\gamma_{JKF}t-\phi(j,k,t)-\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]\right\},$$
$$\cos\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\cos[\gamma_{JKF}t-\phi(j,k,t)]=\frac{1}{2}\left\{\cos\left[\gamma_{JKF}t-\phi(j,k,t)+\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]\right.$$
$$\left.+\cos\left[\gamma_{JKF}t-\phi(j,k,t)-\frac{n\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right]\right\}, \tag{26}$$

Neglecting terms that do not contribute to variational equations, equation (25) reduces to

$$-2\gamma_{JKF}\dot{A}(j,k,t)\sin[\gamma_{JKF}t-\phi(j,k,t)]+2\gamma_{JKF}A(j,k,t)\dot{\phi}(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]$$
$$-2\eta_1^*\gamma_{JKF}^2A(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]-4\eta_1^*\gamma_{JKF}^2A(j,k,t)\sin^2\frac{k\pi y_0}{L_y}\cos[\gamma_{JKF}t-\phi(j,k,t)]$$
$$-2\eta_1^*\frac{c^2j^2\pi^2}{L_x^2}A(j,k,t)\cos[\gamma_{JKF}t-\phi(j,k,t)]-4\eta_1^*\frac{c^2j^2\pi^2}{L_x^2}A(j,k,t)\sin^2\frac{k\pi y_0}{L_y}\cos[\gamma_{JKF}t-\phi(j,k,t)]=0\,. \tag{27}$$

The variational equations, respectively, are

$$-2\gamma_{JKF}\dot{A}(j,k,t)=0 \tag{28}$$

and

$$2\gamma_{JKF}A(j,k,t)\dot{\phi}(j,k,t)-2\eta_1^*\gamma_{JKF}^2A(j,k,t)-4\eta_1^*\gamma_{JKF}^2A(j,k,t)\sin^2\frac{k\pi y_0}{L_y} \tag{29}$$

which when rearranged and solved yields

$$A(j,k,t)=C_{JK}^0 \tag{30}$$

and

$$\phi(j,k,t)=\frac{\eta_1^*}{2}\left\{(2\gamma_{JKF}+4\gamma_{JKF}S_a(k,k))+\frac{(2S_a(j,j)+4S_c(j,k))}{\gamma_{JKF}}\right\}t+\phi_{JK} \tag{31}$$

where

$$S_a(k,k)=\sin^2\frac{k\pi y_0}{L_y},\quad S_b(j,j)=\frac{c^2 j^2\pi^2}{L_x^2}\quad \text{and}\quad S_c(j,k)=\frac{c^2 j^2\pi^2}{L_x^2}\sin^2\frac{k\pi y_0}{L_y}. \tag{31}$$

Therefore, when the mass effect of the particle is considered, the first approximation to the homogeneous system is given by

$$\overline{U}(j,k,t)=C_{JK}^0\cos\left[\gamma_{JKM}t-\phi_{JK}\right], \tag{32}$$

where

$$\gamma_{JKM}=\gamma_{JKF}\left\{1-\frac{\eta_1^*}{2}\left[\left(2+4S_a(k,k)\right)+\frac{\left(2S_a(j,j)+4S_c(j,k)\right)}{\gamma_{JK}^2}\right]\right\}, \tag{33}$$

represents the modified natural frequency representing the frequency of the free system due to the presence of the mass of the particle.

Next, one replaces (17) by the equivalent free system operator defined by the modified frequency γ_{JKM}, i.e

$$\overline{U}_{tt}(j,k,t)+\gamma_{JKM}^2\overline{U}(j,k,t)=\eta_1^* L_x L_y g\sin\frac{k\pi y_0}{L_y}\sin\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right), \tag{34}$$

Clearly, equation (34) is analogous to equation (14). Thus following arguments similar to the previous ones, the solution of the equation (34) is thus obtained as

$$\begin{aligned}
U(x,y,t)=\frac{4}{L_xL_y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\eta_1^*L_xL_yg\sqrt{\pi}}{2\gamma_{JKM}\sqrt{2a_0}}\Bigg(&Sin\gamma_{JKM}t\Bigg[Cos\left(\frac{b_2^2}{4a_0}-c_0\right)S\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)-C\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_2^2}{4a_0}-c_0\right)\\
&+Cos\left(\frac{b_1^2}{4a_0}-c_0\right)S\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)-C\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_1^2}{4a_0}-c_0\right)-Cos\left(\frac{b_2^2}{4a_0}-c_0\right)S\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)\\
&+C\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_2^2}{4a_0}-c_0\right)-Cos\left(\frac{b_1^2}{4a_0}-c_0\right)S\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)+C\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_1^2}{4a_0}-c_0\right)\Bigg]\\
&-Cos\gamma_{JKM}t\Bigg[Cos\left(\frac{b_1^2}{4a_0}-c_0\right)C\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)+S\left(\frac{b_1+2a_0t}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_1^2}{4a_0}-c_0\right)-Cos\left(\frac{b_2^2}{4a_0}-c_0\right)C\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)\\
&-S\left(\frac{b_2+2a_0t}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_2^2}{4a_0}-c_0\right)-Cos\left(\frac{b_1^2}{4a_0}-c_0\right)C\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)-S\left(\frac{b_1}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_1^2}{4a_0}-c_0\right)\\
&+Cos\left(\frac{b_2^2}{4a_0}-c_0\right)C\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)+S\left(\frac{b_2}{\sqrt{2\pi a_0}}\right)Sin\left(\frac{b_2^2}{4a_0}-c_0\right)\Bigg]\Bigg)\left(\sin\frac{k\pi y}{L_y}\sin\frac{j\pi x}{L_x}\right)
\end{aligned} \tag{35}$$

,

which represents the transverse displacement response to concentrated masses moving at variable velocities of a simply-supported isotropic rectangular plate with rotatory inertia correction factor resting on Winkler type elastic foundation.

6 COMMENTS ON CLOSED FORM SOLUTIONS

In an undamped vibrating system such as this, the response amplitude of the structure may grow without bound. When this happens it is called resonance. This is a crucial phenomenon in any engineering design particularly in bridge engineering.

Equation (16) clearly shows that the Simply Supported elastic beam resting on elastic foundation and traversed by moving force reaches a state of resonance whenever

$$\gamma_{JKF} = \frac{j\pi c_c}{L_x}, \; \gamma_{JKF} = \frac{j\pi c_c}{L_x} + 2a_0 t_c, \tag{36}$$

while equation (35) indicates that the same beam under the action of moving mass will experience resonance effect whenever

$$\gamma_{JKM} = \frac{j\pi c_c}{L_x}, \; \gamma_{JKM} = \frac{j\pi c_c}{L_x} + 2a_0 t_c \tag{37}$$

where c_c and t_c are respectively the critical velocity and critical time at which resonance occurs. From equation (33), it is known that

$$\gamma_{JKM} = \gamma_{JKF}\left\{1 - \frac{\eta_1^*}{2}\left[\left(2 + 4S_a(k,k)\right) + \frac{\left(2S_b(j,j) + 4S_c(j,k)\right)}{\gamma_{JKF}^2}\right]\right\} \tag{38}$$

which implies

$$\gamma_{JKF} = \frac{\dfrac{j\pi c_c}{L}}{\left\{1 - \dfrac{\eta_1^*}{2}\left[\left(2 + 4S_a(k,k)\right) + \dfrac{\left(2S_b(j,j) + 4S_c(j,k)\right)}{\gamma_{JKF}^2}\right]\right\}} \tag{39}$$

It is therefore evident, that for the same natural frequency, the critical velocity for the system consisting of a Simply Supported isotropic rectangular plate resting on an elastic foundation and traversed by concentrated forces moving with non-uniform velocities is greater than that of the moving mass problem. Thus, for the same natural frequency of an isotropic rectangular plate, resonance is reached earlier in the moving mass system than in the moving force system.

7 RESULTS AND DISCUSSION

To illustrate the foregoing theories and analysis the example in [27] is adopted and an isotropic plate of modulus of elasticity E = 3.1×10^{10} N/m^2, with dimensions Lx = 100m, Ly=10m, thickness h=0.3m and the poisson ratio $\nu = 0.35$ is considered. Furthermore, the values of foundation moduli are varied between $0\,N/m^3$ and $4000000\,N/m^3$, the values of axial force N_x are varied between 0 N and $2\cdot0\times10^8\,N$.

Figures 1 and 2 depict the transverse displacement response of a simply supported isotropic rectangular plate under the action of concentrated forces moving at variable velocity for various values of axial force N_x and N_y respectively. These figures show that, for fixed values of subgrade moduli K=40000 and Rotatory inertia correction factor Ro=50, as the values of N_x or N_y increases, the dynamic deflection of the isotropic rectangular plate decreases. Similar results are obtained when the simply supported plate is subjected to concentrated masses travelling at variable velocity as shown in figures 5 and 6. For various travelling time t, the deflection profile of the plate under the action of moving forces for various values of subgrade moduli K and for fixed values of axial forces N_x=2000000, N_y=20000 and Rotatory inertia correction factor Ro=50 are shown in figure 3. It is observed that higher values of subgrade moduli K reduce the deflection profile of the vibrating plate structure. The same behaviour characterizes the deflection profile of the simply supported plate under the action of concentrated masses moving at variable velocity for various values of subgrade moduli K as shown in figure 7. Also, figures 4 and 8 display the displacement responses of the simply supported isotropic rectangular plate respectively to concentrated forces and concentrated masses travelling with variable velocity for various values of rotatory inertia Ro and for fixed values of axial forces N_x=20000 00, N_y=20000 and subgrade moduli K=40000. These figures clearly show

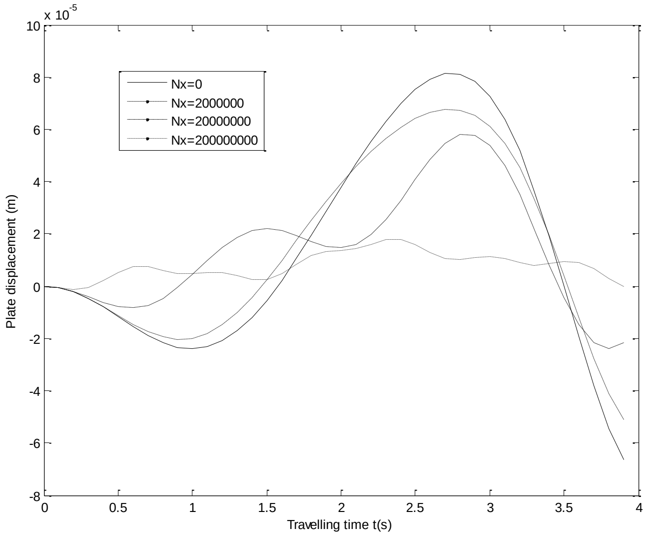

Figure 1: Transverse displacement response of a simply supported isotropic rectangular plate under the actions of concentrated forces travelling at variable velocities for various values of axial force along x direction N_x and for fixed values of foundation constant K=40000, axial force along y N_y=20000 and rotatory inertia Ro=50.

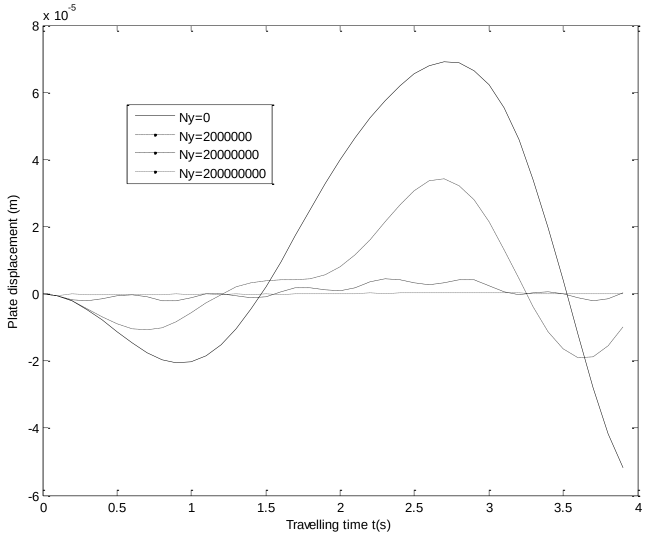

Figure 2: Transverse displacement response of a simply supported isotropic rectangular plate under the actions of concentrated forces travelling at variable velocities for various values of axial force along y direction N_y and for fixed values of foundation constant K=40000, axial force along x N_x=2000000 and rotatory inertia Ro=50.

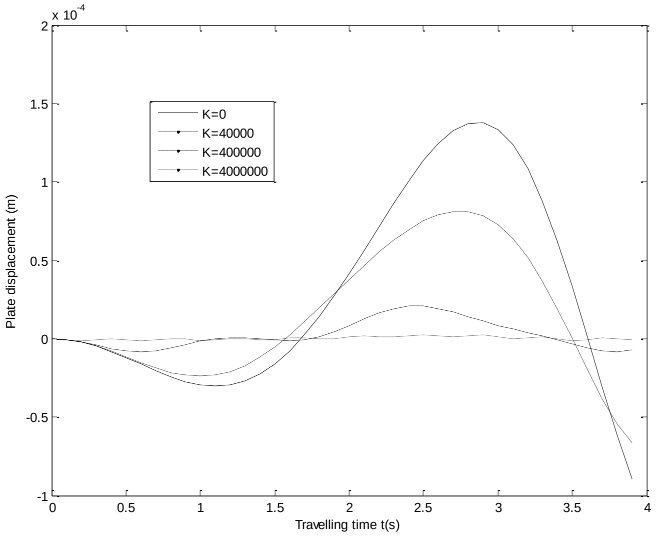

Figure 3: Deflection profile of a simply supported isotropic rectangular plate under the actions of concentrated forces travelling at variable velocities for various values of foundation constant K and for fixed values of axial force N_x=2000000, N_y=20000 and rotatory inertia Ro=50.

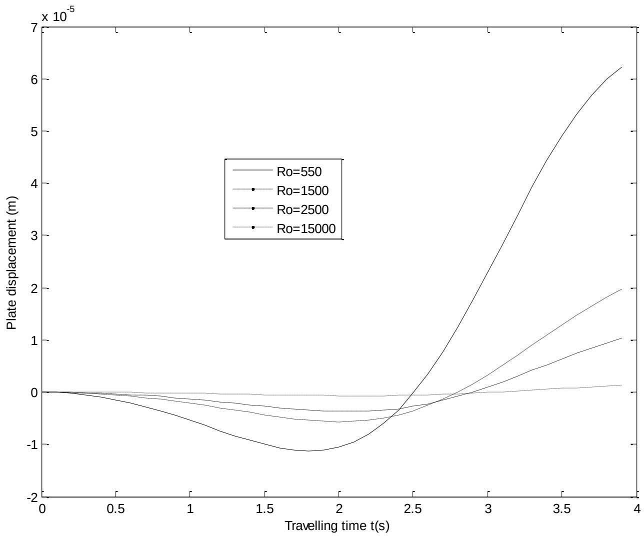

Figure 4: Displacement response of a simply supported isotropic rectangular plate under the actions of concentrated forces travelling at variable velocities for various values of rotatory inertia Ro and for fixed values of foundation constant K =40000, axial force N_x=2000000 and N_y=20000.

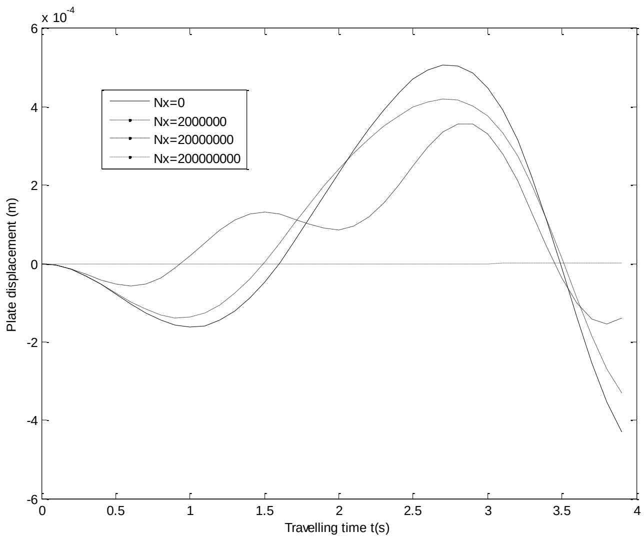

Figure 5: Transverse displacement response of a simply supported isotropic rectangular plate under the actions of concentrated masses travelling at variable velocities for various values of axial force N_x and for fixed values of foundation constant K =40000, N_y=20000 and rotatory inertia Ro=50

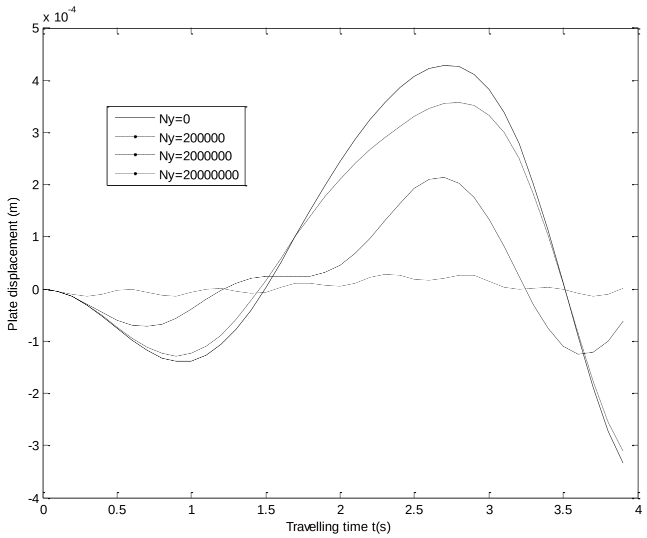

Figure 6: Transverse displacement response of a simply supported isotropic rectangular plate under the actions of concentrated masses travelling at variable velocities for various values of axial force N_y and for fixed values of foundation constant K =40000, N_x=2000000 and rotatory inertia Ro=50.

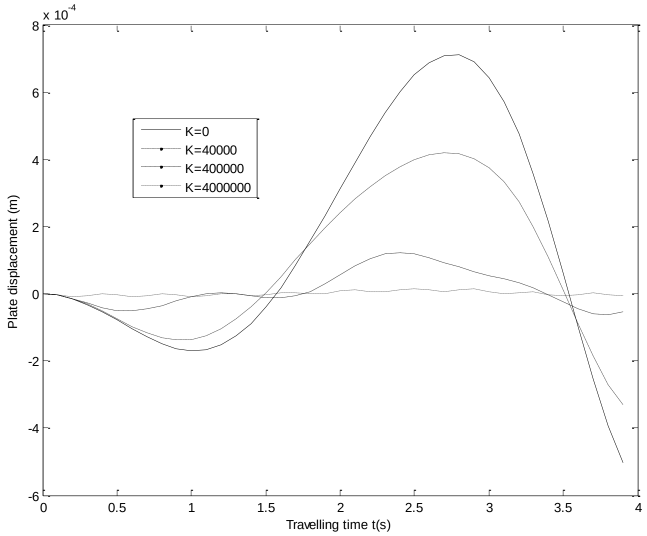

Figure 7: Deflection profile of a simply supported isotropic rectangular plate under the actions of concentrated masses travelling at variable velocities for various values of foundation constant K and for fixed values of axial force N_x=2000000, N_y=20000 and rotatory inertia Ro=50.

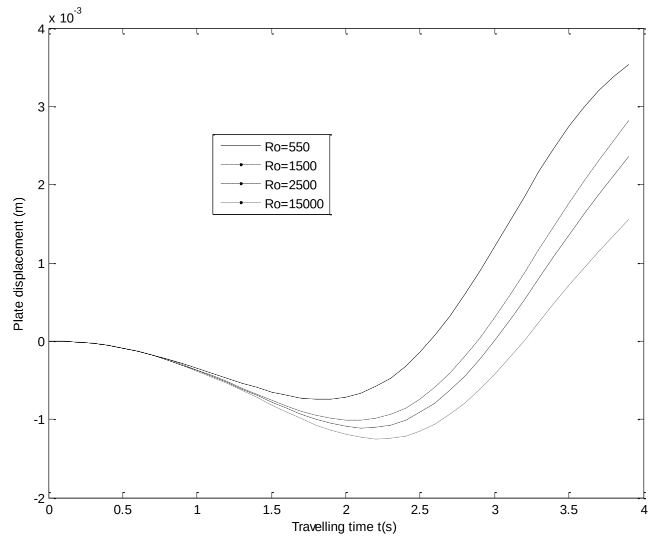

Figure 8: Displacement response of a simply supported isotropic rectangular plate under the actions of concentrated masses travelling at variable velocities for various values of rotatory inertia Ro and for fixed values of foundation constant K =40000, axial force N_x=2000000 and N_y=20000.

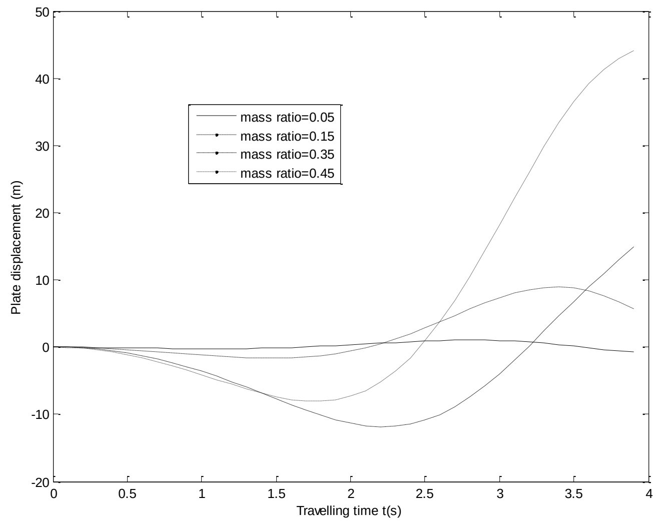

Figure 9: Response amplitudes of a simply supported isotropic rectangular plate under the actions of concentrated masses travelling at variable velocities for various values of the mass ratio η_0 and for fixed values of of N_x=2000000, N_y=20000, K=40000 and Ro=50.

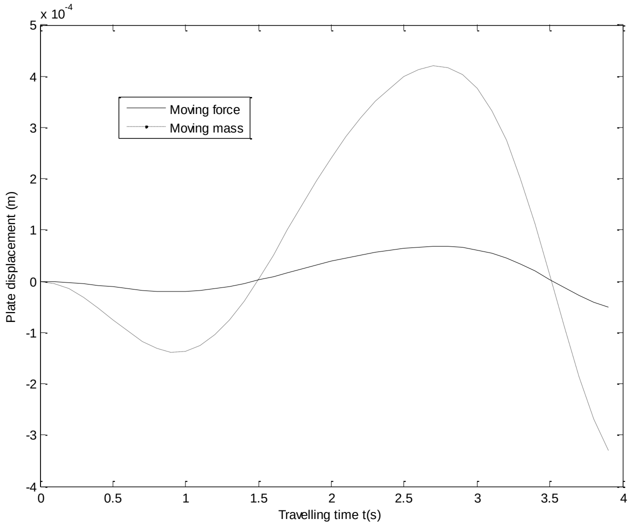

Figure 10: Comparison of the displacement response of moving force and moving mass cases of a simply supported isotropic rectangular plate for fixed values of N_x=2000000, N_y=20000, K=40000 and Ro=50.

that as the values of rotatory inertia correction factor increases, the displacement response of the simply supported plate under the action of both concentrated forces and concentrated masses travelling at variable velocity decreases. Also, for various travelling time t, the response amplitude of the isotropic plate under the action of accelerating traveling masses for fixed values of subgrade moduli K=40000, axial forces N_x=2000000, N_y=20000 and Rotatory inertia correction factor Ro=50 are shown in figure 9. It is observed that larger values of the mass ratio, η_0, increases the response amplitude of the isotropic plated subjected to accelerating masses.

Finally, figure 10. depicts the comparison of the transverse displacement response of moving force and moving mass cases of a simply supported rectangular plate traversed by a moving load travelling at variable velocity for fixed values of N_x=2000000, N_y=20000, K=40000 and Ro=50. Evidently, the response amplitude of the moving mass is higher than that of the moving force. This important result shows that, the moving force solution is not always an upper bound to the solution of the moving mass problem. Hence the inertia of the moving load must always be taken into consideration for accurate and reliable assessment of the response to moving load of elastic structures.

8 CONCLUSION

The problem of the flexural motions of a prestressed isotropic rectangular plate resting on elastic foundation and traversed by concentrated masses traveling with variable velocity has been investigated. Closed form solution of the governing fourth order partial differential equations with variable and singular coefficients of the plate-mass problems is presented. For this two-dimensional dynamical problem, the solution techniques is based on the double Fourier finite sine integral transformation with respect to the spatial variables x and y, the expansion of the Dirac Delta function in series form, a modification of Struble's asymptotic method and the use of Fresnel sine and Fresnel cosine integrals. The solutions so obtained are analyzed and resonance conditions for the various problems solved are established. Numerical calculations show that

I. in the dynamical problem considered, resonance is reached earlier in a system traversed by moving mass than in that under the action of a moving force.

II. as the axial force increases, the response amplitudes of isotropic rectangular plate carrying concentrated loads moving at non-uniform velocities decrease.

III. when the values of axial force and rotatory inertia are fixed, the displacements of isotropic rectangular plate resting on elastic foundation and traversed by masses traveling with variable velocities decrease as the value of foundation moduli K increases for all variants of the boundary conditions.

IV. for fixed values axial force and foundation modulus K, the response amplitude for the moving mass problem is greater than that of the moving force problem for all illustrative end conditions considered.

V. the moving force solution is not always an upper bound for the accurate solution of the moving mass problems in the dynamical system under consideration. Hence, inertia of the moving load must always be considered when assessing the response to moving load of elastic structures as it affects the vibrating system significantly and finally,

VI. for the same natural frequency, the critical velocity for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem.

VII. increased values of all the aformentioned structural parameters increases the dynamic stability of the vibrating system thereby reducing the risk of resonance.

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