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# Static analysis of nanoplates based on the nonlocal Kirchhoff and Mindlin plate theories using DQM

**Abstract**
In this study, static analysis of the two-dimensional rectangular nanoplates are investigated by the Differential Quadrature Method (DQM). Numerical solution procedures are proposed for deflection of an embedded nanoplate under distributed nanoparticles based on the DQM within the framework of Kirchhoff and Mindlin plate theories. The governing equations and the related boundary conditions are derived by using nonlocal elasticity theory. The difference between the two models is discussed and bending properties of the nanoplate are illustrated. Consequently, the DQM has been successfully applied to analyze nanoplates with discontinuous loading and various boundary conditions for solving Kirchhoff and Mindlin plates with small-scale effect, which are not solvable directly. The results show that the above mentioned effects play an important role on the static behavior of the nanoplates.

**Keywords**
Nanoplate, Small-scale effect, Mindlin plate, Kirchhoff plate, Differential Quadrature Method.

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## 1 INTRODUCTION

Nanostructures have significant mechanical, electrical and thermal performances that are superior to the conventional structural materials. They have attracted much attention in modern science and technology. For example, in micro/nano electromechanical systems (MEMS/NEMS), nanostructures have been used in many areas, including communications, machinery, information technology, biotechnology technologies.

So far, three main methods were provided to study the mechanical behaviors of nanostructures. These are atomistic model (Ball, 2001; Baughman *et al.*, 2002), semi-continuum (Li and Chou, 2003) and continuum models (Govindjee and Sackman, 1999; He *et al.*, 2005). However, both atomistic and semi-continuum models are computationally expensive and are unsuitable for analyzing large-scale systems. On the other hand, studying the vibration of nanostructures is important in nanotechnology. Understanding the static behavior of nanostructures is a key step for

MEMS/NEMS device design. There have been some studies on the vibration behavior and buckling of nanostructures using the continuum model (Kitipornchai *et al.*, 2005; Akhavan *et al.*, 2009).

The basic idea of the differential quadrature method lies in the approximation of partial derivative of a function with respect to a coordinate at a discrete point as a weighted linear sum of the function values at all discrete points along that coordinate direction. DQM has been found to be an efficient numerical technique for the solution of initial and boundary value problems. The DQ technique has been widely used for solving various dynamic and stability problems of large-scale structures (Nikkhoo *et al.*, 2012; Nikkhoo and Kananipour, 2014) and small-scale too (Kananipour *et al.*, 2014; Mohammadi *et al.*, 2013; Mohammadi *et al.*, 2014).

In the present paper, the static response of an embedded nanoplate with simply-supported and clamped under distributed nanoparticles is studied based on the DQM. A detailed parametric study is conducted to study the influences of the material length scale parameter, the nonlocal elasticity factors, the various boundary conditions and the elastic medium constant as well as the solution procedures on the static responses of the nanoplate. From the literature survey, it is found that the effect of nonlocal elasticity on the static behavior of nano-scale plates has been investigated. Both Kirchhoff and Mindlin plate theories will be discussed. The effects of nonlocal parameter and transverse shear deformation of the plate on the bending deflection of the plate are studied for different values of the plate size.

## 2 PROBLEM FORMULATION

### 2.1 Nonlocal elasticity

In this investigation, a double-layered graphene sheet is modeled as a rectangular plate with thickness h, length $L_x$ and width $L_y$, which located on the elastic foundation. A Cartesian coordinate system (x, y, z) is used for nanoplate, see Figure 1, with the x, y and z axes along the length, width and thickness of nanoplate, respectively.

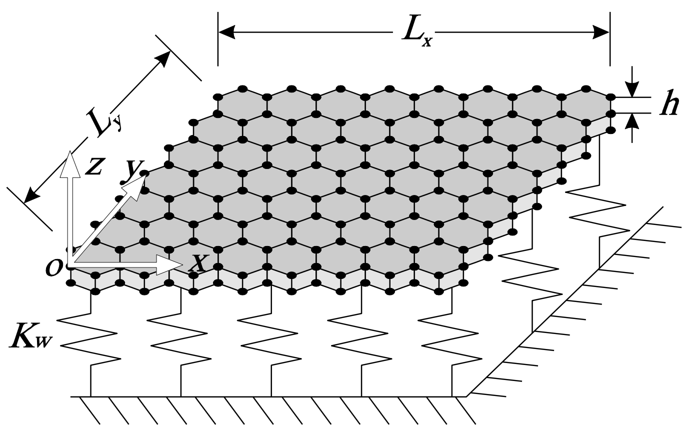


**Figure 1**: Problem geometry of a double-layered graphene sheet.

According to the nonlocal continuum theory (Eringen, 1972; Eringen, 1983), the stress at a reference point depends on strain at all points in the body. The nonlocal constitutive equations can be simplified to

$$(1-(e_0a)^2\nabla^2)\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{1}$$

where $\sigma_{ij}$, $C_{ijkl}$ and $\varepsilon_{kl}$ are the nonlocal elasticity stress tensor, fourth order local stress tensor and strain tensor, respectively. The parameter $e_0$ is estimated nonlocal elasticity constant suitable to each material, and a is the internal characteristic length (e.g. the C-C bond length, lattice parameter and granular size). Furthermore, $e_0a$ is nonlocal parameter or distinctive length that means the scale coefficient which denotes the small-scale effect on the mechanical characteristics. Choice of the value of a parameter $e_0$ is crucial to calibrate the nonlocal model with experimental results. Eringen (1972) determined a value of 0.39 for this parameter by matching the dispersion curves based on atomic models. Sudak (2003) used the length of C–C bond equal to 0.142 nm for carbon nanotubes (CNTs) stability analysis as internal characteristic length a. Wang and Hu (2005) used strain gradient method to propose an estimate of the value around $e_0=0.288$. In the limit when $e_0a$ goes to zero, nonlocal elasticity will be reduced to the classical local mode. Generally, for the analysis of carbon nanoplates, the nonlocal scale coefficients $e_0a$ are taken in the range 0–2 nm (Wang and Wang, 2007). Still contemporary research is going on to find the exact values of nonlocal parameters for various nanolevel structural problems (Murmu and Pradhan, 2009). Furthermore, $\nabla^2$ is the Laplacian operator and is given by $\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.

## 2.2 Mindlin plate theory

The displacement field with the effect of the transverse shear and rotary inertia can be expressed as

$$u_x = u(x,y,z) + z\psi_x, \tag{2.a}$$

$$u_y = v(x,y,z) + z\psi_y, \tag{2.b}$$

$$u_z = w(x,y,z), \tag{2.c}$$

where $\psi_x$ and $\psi_y$ are the local rotations for the x and y direction, respectively. Using Eq. (1) and according to Hook's law, the plane stress nonlocal constitutive relations can be expressed as

$$(1-(e_0a)^2\nabla^2)\sigma_{xx} = \frac{E}{1-\upsilon^2}(\varepsilon_{xx} + \upsilon\varepsilon_{yy}), \tag{3.a}$$

$$(1-(e_0a)^2\nabla^2)\sigma_{yy} = \frac{E}{1-\upsilon^2}(\varepsilon_{yy} + \upsilon\varepsilon_{xx}), \tag{3.b}$$

$$(1-(e_0a)^2\nabla^2)\sigma_{yz} = 2G\varepsilon_{yz}, \tag{3.c}$$

$$(1-(e_0a)^2\nabla^2)\sigma_{xz} = 2G\varepsilon_{xz}, \tag{3.d}$$

$$(1-(e_0a)^2\nabla^2)\sigma_{xy} = 2G\varepsilon_{xy}. \tag{3.e}$$

Here E, G and υ are the Young's modulus, shear modulus equal to E/2(1+υ) and poison's ratio, respectively. Furthermore, the general strains can be expressed as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + z\frac{\partial \psi_x}{\partial x}, \tag{4.a}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + z\frac{\partial \psi_y}{\partial y}, \tag{4.b}$$

$$\varepsilon_z = 0, \tag{4.c}$$

$$\varepsilon_{xz} = \frac{1}{2}(\frac{\partial w}{\partial x} + \psi_x), \tag{4.d}$$

$$\varepsilon_{yz} = \frac{1}{2}(\frac{\partial w}{\partial y} + \psi_y), \tag{4.e}$$

$$\varepsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\frac{\partial \psi_x}{\partial y} + z\frac{\partial \psi_y}{\partial x}). \tag{4.f}$$

From Eqs. (3) and (4), the nonlocal shear force and moment resultants become

$$T = \left\{T_{xx}, T_{yy}, T_{xy}\right\}^T = \int_{-h/2}^{h/2} \left\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\right\}^T dz, \tag{5.a}$$

$$M = \left\{M_{xx}, M_{yy}, M_{xy}\right\}^T = \int_{-h/2}^{h/2} \left\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\right\}^T z\,dz, \tag{5.b}$$

which leads to

$$(1-(e_0a)^2\nabla^2)M_{xx} = D(\frac{\partial \psi_x}{\partial x} + \upsilon\frac{\partial \psi_y}{\partial y}), \tag{6.a}$$

$$(1-(e_0a)^2\nabla^2)M_{yy} = D(\frac{\partial \psi_y}{\partial y} + \upsilon\frac{\partial \psi_x}{\partial x}), \tag{6.b}$$

$$(1-(e_0a)^2\nabla^2)M_{xy} = \frac{D(1-\upsilon)}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right), \tag{6.c}$$

$$(1-(e_0a)^2\nabla^2)T_x = \kappa Gh\left(\frac{\partial w}{\partial x}+\psi_x\right), \tag{6.d}$$

$$(1-(e_0a)^2\nabla^2)T_y = \kappa Gh\left(\frac{\partial w}{\partial y}+\psi_y\right). \tag{6.e}$$

where $M_{xx}$ and $M_{yy}$, $M_{xy}$, $T_x$ and $T_y$ are bending moments, twisting moment and shear forces per unit of length. In which $D=Eh^3/12(1-\nu^2)$ is flexural rigidity and $\varkappa$ the shear correction factor. Calculation of the shear correction coefficient can be performed by using various methods. Some approaches have been discussed in (Vlachoutsis, 1992; Rikards *et al.*, 1994; Altenbach, 2000). Further in numerical examples the shear correction factor has the value $10(1+\upsilon)/(12+11\upsilon)$. The governing equations based on the Mindlin plate theory with external load, q(x, y), are given as (Mindlin, 1951)

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y} = T_x, \tag{7.a}$$

$$\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x} = T_y, \tag{7.b}$$

$$\frac{\partial T_x}{\partial x}+\frac{\partial T_y}{\partial y} = -q+k_w w-G_b\nabla^2 w, \tag{7.c}$$

where $k_w$ the Winkler foundation modulus, $G_b$ the stiffness of the shearing layer.
Based on Eqs. (6) and Eqs. (7), the governing equations of Mindlin plate with small-scale effect can be derived as

$$\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}-\frac{\kappa Gh}{D}\left(\frac{\partial w}{\partial x}+\psi_x\right)=0, \tag{8.a}$$

$$\frac{\partial^2\psi_y}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}-\frac{\kappa Gh}{D}\left(\frac{\partial w}{\partial y}+\psi_y\right)=0, \tag{8.b}$$

$$\nabla^2 w+\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}\right)=-\frac{(1-(e_0a)^2\nabla^2)}{\kappa Gh}(q+G_b\nabla^2 w-k_w w). \tag{8.c}$$

## 2.3 Kirchhoff plate theory

If the shear forces and rotational effect are not considered, the results of a nonlocal Mindlin plate model will be reduced to the nonlocal Kirchhoff plate model, and the governing equations can be given as

$$D\nabla^2\nabla^2 w = (1-(e_0a)^2\nabla^2)(q + G_b\nabla^2 w - k_w w). \tag{9}$$

## 2.4 Boundary Conditions

In generally, various types of support conditions are similar to table 1.

| Edge condition | Prescribed D.O.F | Natural condition |
|---|---|---|
| Clamped | w= $\vartheta_n$= $\vartheta_p$=0 | None |
| Simply supported | w=0 | $M_n$=0 |
| Free | None | T=$M_n$=$M_p$=0 |

$M_n$, $\vartheta_n$, - rotation and moment normal to edge
$M_p$, $\vartheta_p$ - rotation and moment perpendicular to edge

**Table 1:** Support conditions.

We will assume simply supported and clamped boundary conditions along all the four edges of the graphene sheets. In Figure 2, Two types of the boundary conditions are considered.

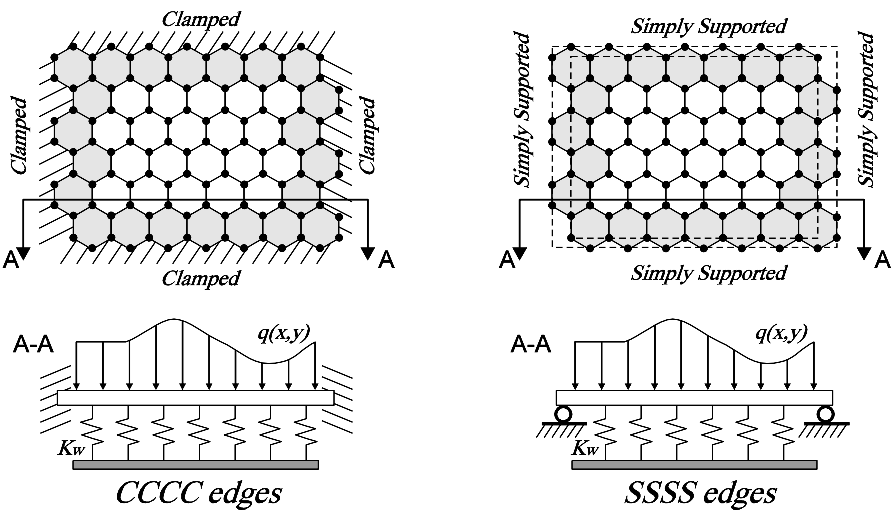


**Figure 2**: Simply supported and clamped boundary conditions.

Herein, the boundary conditions are mathematically written as

1. For Mindlin plate model, the boundary conditions are

- All edges simply supported (SSSS)

$$\begin{aligned} &w=0, \quad M_{xx}=0, \quad \psi_y=0, \quad at\ edges \quad x=0,L_x \\ &w=0, \quad M_{yy}=0, \quad \psi_x=0, \quad at\ edges \quad y=0,L_y \end{aligned} \tag{10.a}$$

- All edges clamped (CCCC)

$$w=0, \quad \psi_x=0, \quad \psi_y=0, \quad at\ edges \quad x=0,L_x\ and\ y=0,L_y \tag{10.b}$$

2. For Kirchhoff plate model, the boundary conditions are

- All edges simply supported (SSSS)

$$\begin{aligned} &w=0, \quad \frac{\partial^2 w}{\partial x^2}=0, \quad at\ edges \quad x=0,L_x \\ &w=0, \quad \frac{\partial^2 w}{\partial y^2}=0, \quad at\ edges \quad y=0,L_y \end{aligned} \tag{11.a}$$

- All edges clamped (CCCC)

$$\begin{aligned} &w=0, \quad \frac{\partial w}{\partial x}=0, \quad at\ edges \quad x=0,L_x \\ &w=0, \quad \frac{\partial w}{\partial y}=0, \quad at\ edges \quad y=0,L_y \end{aligned} \tag{11.b}$$

## 3 DIFFERENTIAL QUADRATURE PROCEDURE

Many researchers have recently suggested the application of the differential quadrature (DQ) method to the analysis of nanostructures (Khodami Maraghi *et al.*, 2013; Farajpour *et al.*, 2013; Ghorbanpour Ansari *et al.*, 2013; Mousavi *et al.*, 2013) as an accuracy, efficiency and great potential in solving complicated partial differential equations. DQ is capable of calculating derivative orders of the field variable up to N-1 order in the case of N grid points. DQ equations based on polynomial or Fourier's series expansions are computable; in this paper, DQ based on polynomials, which provides fine compatibility in analyzing high-order differential equations, is employed. A test function is required for deriving DQ equations; moreover, Shu (2000) proved the Lagrange interpolation polynomials as the test function generates the best convergence. It assumed one-dimensional function, which $w(x_k)$ are field variables at the point $x_k$ (k=1,2,...,N). The first-order derivative for the function w(x) at the ith grid point is calculated via summing weighting-linear function values in the other nodes (Eq. (12)). Conveniently, nth order derivative (n=2, 3, ..., N-1) at the ith grid point can be calculated in the same way (Eq. (13))

$$\frac{dw(x_i)}{dx} = \sum_{k=1}^{N} c_{ik}^{(1)} w(x_k), \tag{12}$$

$$\frac{d^n w(x_i)}{dx^n} = \sum_{k=1}^{N} c_{ik}^{(n)} w(x_k), \tag{13}$$

where N is the number of grid points in the x-direction, $c_{ik}^{(1)}$ and $c_{ik}^{(n)}$ are the weighting coefficient associated with the first and nth-order partial derivative of w(x) with respect to x at the discrete point $x_i$.

Weighting coefficients for the first and nth-order derivative are obtained from the following recurrence equations

$$c_{ik}^{(1)} = \frac{R^{(1)}(x_i)}{(x_i - x_k) R^{(1)}(x_k)} \quad i \neq k, \quad i,k = 1,2,\ldots,N, \tag{14}$$

$$c_{ik}^{(n)} = n\left(c_{ii}^{(n-1)} c_{ik}^{(1)} - \frac{c_{ik}^{(n-1)}}{x_i - x_k}\right) \quad i \neq k, \quad n = 2,3,\ldots,N-1, \quad i,k = 1,2,\ldots,N, \tag{15}$$

$$c_{ii}^{(n)} = -\sum_{k=1, k\neq i}^{N} c_{ik}^{(n)} \quad n = 1,2,\ldots,N-1, \quad i = 1,2,\ldots,N, \tag{16}$$

where R(x) and $R^{(1)}$(x) are defined as

$$R(x) = (x - x_1).(x - x_2)\cdots(x - x_N), \tag{17}$$

$$R^{(1)}(x_i) = \prod_{k=1,\, k\neq i}^{N} (x_i - x_k). \tag{18}$$

In which, $x_1$, $x_2$, ..., $x_N$ are coordinated of the grid points that might be selected as desired. Obviously, weighting coefficients of the second and higher-order derivatives is calculable via weighting coefficients of the first-order derivative (Eqs. (14-16)). It has proven the weighting coefficients in multi-dimensional cases similar to one-dimensional case are determinable separately in any directions (Shu, 1991).

## 4 NUMERICAL RESULTS AND DISCUSSION

In this section, numerical calculations of the bending behaviors with the small scale effects are performed. The material constants used in the calculation are defined on table 2.

| | |
|---|---|
| Young's modulus (E) | 1 TPa |
| Poisson ratio ($\nu$) | 0.16 |
| Density ($\rho$) | 2250 kg/m$^3$ |
| Thickness (h) | 0.34 nm |
| Winkler foundation modulus ($K_w$) | 1 kg/m$^2$ |
| Stiffness of the shearing layers ($G_b$) | 1 kg/m |
| $L_x$=$L_y$ | Suppose 10 nm |

**Table 2**: Material properties.

Moreover, the scale coefficient $e_0a$ = 0–2 nm.

The relation between displacement ratio and the nonlocal scale coefficients are presented in Figure 3. Similar relation can be observed, but the square nanoplate are influenced by the scale coefficient significantly. It can be concluded that the small scale effects are obvious on bending properties of the nanoplate.

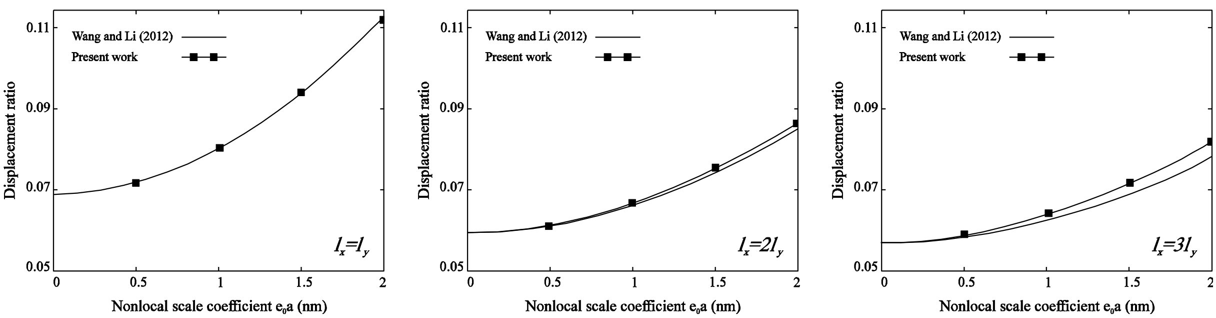


**Figure 3**: Relation between the displacement ratio and the nonlocal scale coefficients ($e_0a$) for the influence of width ratios ($l_x/l_y$).

It can be observed that the displacements ratios decrease quickly when the width ratio ($l_x/l_y$) is smaller than 3. Then, all of the displacement ratios in Figure 3 tend to be three different constants with respective to the nonlocal scale coefficients.

At last, the effects of the elastic matrix are investigated. The relation between the displacement ratio and the width ratio ($l_x/l_y$) with the influences of the Winkler foundation modulus ($k_w$) and the stiffness of the shearing layer ($G_b$) are shown in Figure 4. The nonlocal scale coefficient $e_0a$=1 nm. It can be observed that bigger values of both the Winkler foundation modulus and the stiffness of the shearing layer will result in the larger displacement ratios.

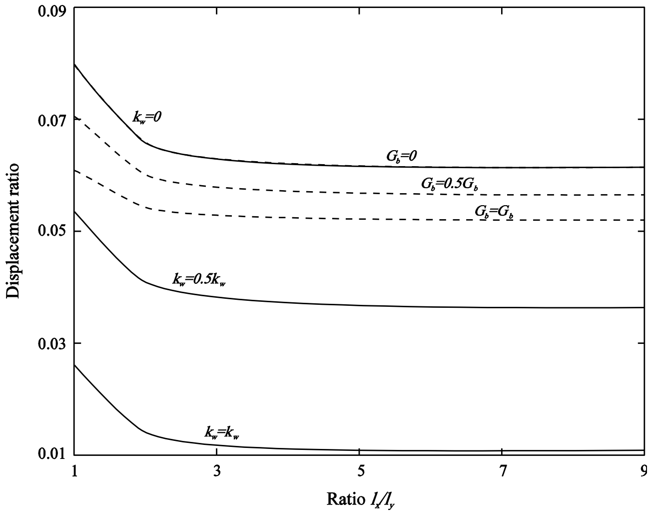


**Figure 4**: Relation between the displacement ratio and the width ratio ($l_x/l_y$) for the influence of Winkler foundation modulus ($k_w$) and stiffness of the shearing layer ($G_b$).

Moreover, this effect is more significant for small width ratios, which is similar to Figure 4 (Wang and Li, 2012).

## 5 CONCLUSIONS

In this research work, bending behaviors of the nanoplate subjected to different in-plane loads were investigated on the basis of the small-scale effects which considered by the nonlocal continuum theory. The governing equations and displacements for the nonlocal Mindlin and Kirchhoff plate models are derived. The influence of the plate models, scale coefficients and width ratios are discussed. From the results, it can be concluded that nonlocal Mindlin plate model is more proper for the thick nanoplate. The displacement ratio becomes larger with the Winkler foundation modulus and the stiffness of the shearing layer increasing.

Furthermore, the two-dimensional differential quadrature method (DQM) has been developed for the bending analysis of nonlocal Mindlin and Kirchhoff plates by integrating the domain decomposition method with the DQ method. Consequently, this method has been successfully applied to the analysis of nanoplates with discontinuities in loading, geometry and boundary conditions. It is hoped that this work can present an effective model to design and analyze the mechanical properties of nanoscale devices.

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