Procedures for Teaching Variational Formulation and Finite Element Approximation of Mechanical Problems applied to the Kirchhoff Plate Model

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July 27, 2004

Abstract

Traditionally, engineering students have been taught Mechanics of Solids and Strength of Materials through a technical theory based on one-dimensional models of bars and beams. Most of textbooks are organized in chapters which present the main concepts of mechanics (equilibrium, strain, stress, etc) for specific problems with no relation to the general case. In addition, the features of the mechanical models of beam and bar are discussed in many chapters and students do not have an overall view of these models.

Due to the multidisciplinary characteristic of the real engineering problems and the intensive use of computers to simulate their behaviour, it is very important to present the formulation and approximation of mechanical models using procedures which make clear all the relevant aspects, hyphoteses and limitations. This paper presents two step-by-step procedures for the variational formulation and finite element approximation of mechanical models which are applied to the Kirchhoff plate model.

Keywords:
Plates. Kirchhoff. Finite Element Method (FEM). Variational Formulation. Virtual Work Principle.

1 Introduction

The increasing complexity and multidisciplinary feature of the real engineering problems and the intensive use of computers to simulate their behaviour have demanded a strong background in basic concepts of mechanics to the engineering students. The traditional approach to present such concepts from particular cases should be substituted to another one which allow engineering students to have an increasing and stimulating learning curve. Based on that, teaching should use the current background of students and at the same time enlarge it through the systematic presentation of mechanical models and underlying concepts.

Most of the textbooks currently used in teaching Solid Mechanics are organized in terms of the main concepts of mechanics, i.e., forces, stress, strain, equilibrium, etc. This approach makes difficulty to acquire a complete view of the most commom considered mechanical models of bar (traction), torsion and beam (bending and shear). It seems that the approach to present all features of each mechanical

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model using the same procedure and an increasing complexity of models would contribute significantly to the learning process.

The traditional books of Strength of Materials [1, 2] address the classical theory based on one-dimensional models of bars and beams. In spite of the relative success in the treatment of these problems, this approach does not allow students to recognize clearly the primal and dual variables, the relations between them, the hypotheses associated to the model and how to manage the hypotheses in order to deal with commonly practical engineering problems which are modelled using plane stress, plate, shell and solid models.

A similar characteristic is presented in the most used finite element textbooks. In general, the boundary between the mechanical model and its finite element approximations is not very clear.

This paper presents two procedures to obtain the formulation of mechanical models and their finite element approximations.

The first procedure is based on the variational formulation [3–9] and has the following main steps: description of the geometric hypotheses; definition of the kinematics; determination of the strain components compatible with the defined kinematics; determination of the rigid-body displacements; determination of the internal and external loads using the work concept; application of the Virtual Work Principle to obtain the equilibrium Boundary Value Problem (BVP); use of the material constitutive law to obtain the stress distributions and the BVP in terms of the kinematics. The finite element approximation procedure uses the following steps: description of the strong form; determination of the weak form; approximation of the weak form; local approximation and definition of the finite elements.

The two procedures will be applied to the Kirchhoff plate model [10–13] which has been used to study many engineering problems. The knowledge of the main features of plate models is important to the engineer. It is expected that the application of the two procedures will clarify many points presented in [8, 14]. The model formulation will not use tensors but cartesian components of the displacements, strain and stresses. This choice was motivated aiming to an intuitive presentation of the Kirchhoff plate model.

2 Formulation of the Kirchhoff Model

2.1 Geometric Hypotheses

Plate models are used to study structural components which are submitted to bending loads and have the thickness smaller than the others dimensions. This last characteristic allows to represent the plate using the reference middle surface (see Fig.1(a)). Therefore the geometric domain used for the formulation of plate models is the middle surface.

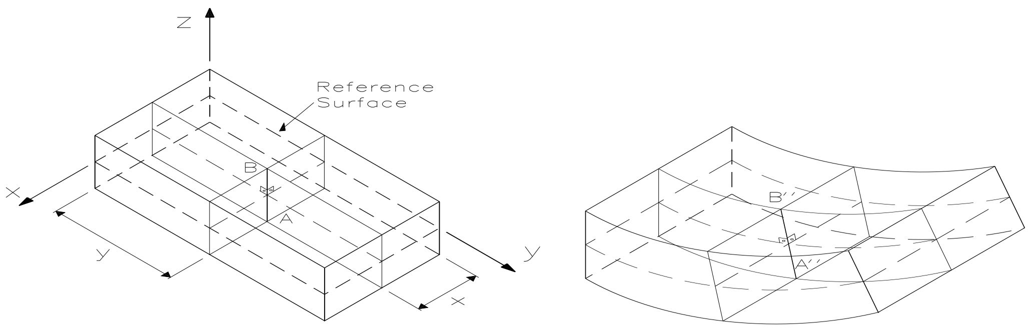
2.2 Definition of the Kinematics

The cartesian reference system and the middle surface used for the formulation of the Kirchhoff plate model are showed in Fig.1(a). It is assumed the case of small displacements.

The plate kinematics consist of bending displacement actions. For the Kirchhoff model, the normal vectors to the undeformed reference surface remain normal to the deformed reference surface and their lengths do not change. Therefore, the transversal shear deformation is null.

The bending kinematics of the Kirchhoff plate is illustrated in Fig.1. Consider the straight line AB normal to the reference surface and located x and y units from the origin of the cartesian reference system origin showed in Fig. 1(a). Due to the plate bending, the normal straight line AB rotates about the x and y axes. It remains normal to the deformed reference surface as indicated by the line A"B" in Fig.1(b).

Specifically, the points of the line AB undergoes a rigid displacement w in the z-direction. After that, AB rotates about the x and y axes and remains normal to the deformed reference surface. As showed

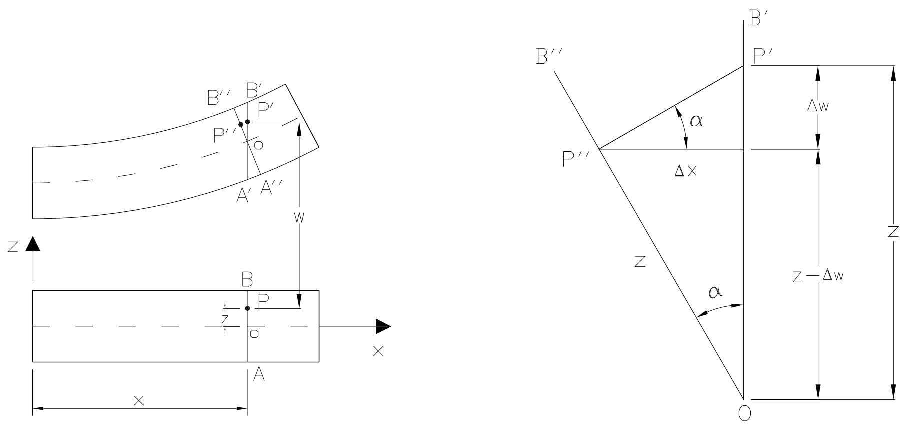


(a) Straight line AB normal to the undeformed reference surface.

(b) Straight line A'B' normal to the deformed reference surface.

Figure 1: Bending kinematics of the Kirchhoff model.

in Fig. 2 for the xz-plane, due to the displacement $w\,(x, y)$, the normal AB reaches the intermediary position A'B'. The line A'B' rotates rigidly of angles α and β, respectively, about the x and y axes and the final position A"B" is normal to the deformed reference surface.



(a) Plane xz.

(b) Detailed kinematics of point P.

Figure 2: Kinematics of the Kirchhoff plate in the xz-plane.

Consider any point P in the normal AB with initial coordinates (x, y, z). After the plate bending, the final position P" is $(x - \Delta x, y - \Delta y, z + w)$ as illustrated in Fig. 2(a). Therefore, in addition to the transversal displacement w, point P has displacement components u and v, respectively, in the x and y-directions. These displacements are given by the difference between the final and initial positions of the point P, i.e.,

$$u(x, y, z) = (x - \Delta x) - x = -\Delta x, \qquad v(x, y, z) = (y - \Delta y) - y = -\Delta y. \tag{1}$$

As the length of the normal AB does not change, the lengths of the segments OP' and OP" showed in Fig.2(a) are z units. Therefore, the following trigonometric relations are valid

$$\sin \alpha = \frac{\Delta x}{z}, \qquad \tan \alpha = \frac{\Delta w}{\Delta x}. \tag{2}$$

For small displacements, $\sin\alpha \approx \alpha$, $\tan\alpha \approx \alpha$ and Δx is small. From equation (2), the angle α is expressed by

$$\alpha = \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = \frac{\partial w(x,y)}{\partial x}. \tag{3}$$

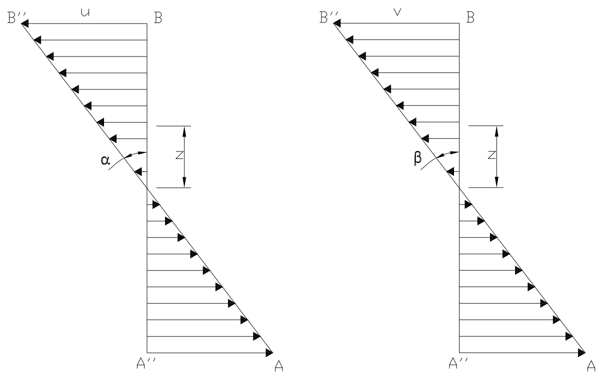
Substituting (2) and (3) into equation (1) and considering $\sin\alpha \approx \alpha$, it follows that

$$u\,(x,y,z) = -z\alpha = -z\frac{\partial w(x,y)}{\partial x}. \tag{4}$$

Analogously, the displacement v is obtained as [15]

$$v\,(x,y,z) = -z\frac{\partial w(x,y)}{\partial y}. \tag{5}$$

From (4) and (5), it may be observed the linear variation of the displacements u and v with the initial coordinate z as illustrated in Fig. 3.



(a) $\alpha = \frac{\partial w}{\partial x} > 0$. (b) $\beta = \frac{\partial w}{\partial y} > 0$.

Figure 3: Linear variation of displacements u and v for the Kirchhoff plate.

In addition to the transversal displacement w, the points of the reference surface may have longitudinal displacements $u_0\,(x,y)$ and $v_0\,(x,y)$, which are known as *membrane displacements*. Due to the assumption of small displacements, the longitudinal displacements in the xy-plane are independent of the transversal displacements w.

Finally, the kinematics of the Kirchhoff plate model for the membrane and bending effects may be expressed by the following displacement vector field

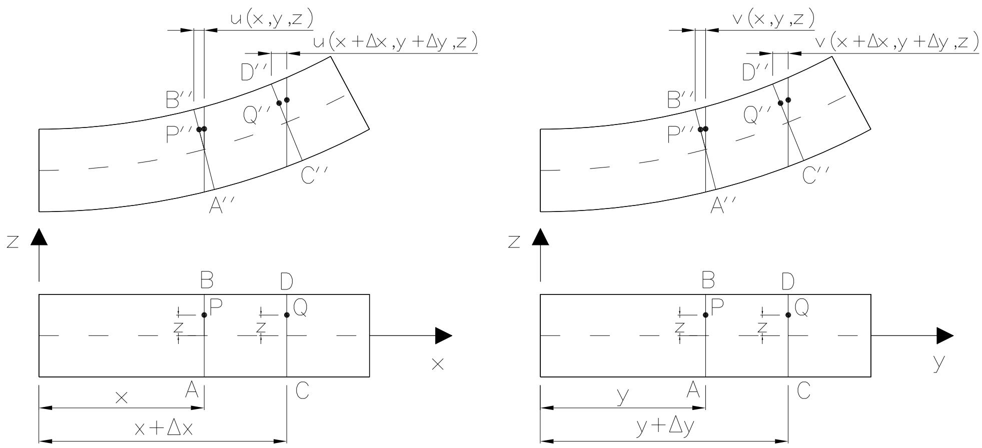
$$\mathbf{u}(x,y,z) = \left\{ \begin{array}{c} u\,(x,y,z) \\ v\,(x,y,z) \\ w\,(x,y) \end{array} \right\} = \left\{ \begin{array}{c} u_0\,(x,y) - z\dfrac{\partial w\,(x,y)}{\partial x} \\ v_0\,(x,y) - z\dfrac{\partial w\,(x,y)}{\partial y} \\ w\,(x,y) \end{array} \right\}. \tag{6}$$

2.3 Strain

Strain is the specific measure of the relative displacement between two points of the body which are at an infinitesimal distance.

The strain state at each point of the plate must be compatible with the kinematics given in equation (6). Consider the points P and Q of two arbitrary normals AB and CD which are at initial positions (x, y, z) and $(x + \Delta x, y + \Delta y, z)$ as illustrated in Fig. 4. The strain measure will be determinated based on the relative displacements of points P and Q.

Observe that the points P and Q have the same coordinate z before the plate bending. The displacements components in the x, y, z-directions of points P and Q are, respectively, $(u(x, y, z), v(x, y, z), w(x, y))$ and $(u(x + \Delta x, y + \Delta y, z), v(x + \Delta x, y + \Delta y, z), w(x + \Delta x, y + \Delta y))$. Their final positions are P" $(x - u(x, y, z), y - v(x, y, z), z + w(x, y))$ and Q" $(x + \Delta x - u(x + \Delta x, y + \Delta y, z), y + \Delta y - v(x + \Delta x, y + \Delta y, z), z + w(x + \Delta x, y + \Delta y))$.



(a) Plane xz. (b) Plane yz.

Figure 4: Relative displacements between points P and Q of the normals AB and CD.

The relative displacement Δu between the points P and Q in the direction x may be obtained from Fig. 4(a) which shows the xz-plane before and after the plate bending. According to the kinematics given in (6), the displacements $u(x, y, z)$ and $u(x + \Delta x, y + \Delta y, z)$ of the points P and Q depend on the transversal displacements $w(x, y)$ and $w(x + \Delta x, y + \Delta y)$, respectively. Hence, from equation (6)

$$\begin{aligned}\Delta u &= u(x + \Delta x, y + \Delta y, z) - u(x, y, z) \\ &= [u_0(x + \Delta x, y + \Delta y) - u_0(x, y)] - z\frac{\partial}{\partial x}[w(x + \Delta x, y + \Delta y) - w(x, y)] \\ &= \Delta u_0 - z\frac{\partial}{\partial x}(\Delta w). \end{aligned} \tag{7}$$

Dividing Δu by the initial distance Δx in the x-direction between points P and Q gives the adimensional relative displacement

$$\frac{\Delta u}{\Delta x} = \frac{\Delta u_0}{\Delta x} - z\frac{\partial}{\partial x}\left(\frac{\Delta w}{\Delta x}\right).$$

The strain component ε_{xx} of the point $P(x, y, z)$ is obtained taking the limit of the previous relation for $\Delta x, \Delta y \to 0$. Hence,

$$\varepsilon_{xx}(x, y, z) = \lim_{\Delta x,\, \Delta y \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u_0(x, y)}{\partial x} - z\frac{\partial^2 w(x, y)}{\partial x^2}. \tag{8}$$

Equation (8) may be rewritten as the sum of the membrane and bending normal strain components, i.e.,

$$\varepsilon_{xx}\left(x,y,z\right)=\varepsilon_{xx}^{m}\left(x,y\right)+\varepsilon_{xx}^{b}\left(x,y,z\right), \tag{9}$$

where

$$\varepsilon_{xx}^{m}\left(x,y\right)=\frac{\partial u_0(x,y)}{\partial x} \quad \text{and} \quad \varepsilon_{xx}^{b}\left(x,y,z\right)=-z\frac{\partial^2 w\left(x,y\right)}{\partial x^2}. \tag{10}$$

The same procedure and Fig. 4(b) may be used to obtain the normal strain component in the y direction. Therefore

$$\varepsilon_{yy}\left(x,y,z\right)=\varepsilon_{yy}^{m}\left(x,y\right)+\varepsilon_{yy}^{b}\left(x,y,z\right), \tag{11}$$

where the membrane and bending components are, respectively,

$$\varepsilon_{yy}^{m}\left(x,y\right)=\frac{\partial v_0(x,y)}{\partial y} \quad \text{and} \quad \varepsilon_{yy}^{b}\left(x,y,z\right)=-z\frac{\partial^2 w\left(x,y\right)}{\partial y^2}. \tag{12}$$

As the transversal displacement $w\left(x,y\right)$ is constant for all points of any normal line to the reference surface, the normal strain $\varepsilon_{zz}\left(x,y\right)=\dfrac{\partial w\left(x,y\right)}{\partial z}$ is zero. This result agrees with the assumption that the lengths of the normals to the reference surface do not change.

According to the kinematics of the Kirchhoff plate, the normal AB remains normal to the deformed reference surface. Based on that, the total transversal shear components is null. Hence, $\bar{\gamma}_{xz}$ and $\bar{\gamma}_{yz}$ are zero.

The total shear strain component in the plane xy for any point P_1 of the reference surface (see Fig. 5) is given by

$$\bar{\gamma}_{xy}=\gamma_1+\gamma_2. \tag{13}$$

The following trigonometric expressions are obtained from Fig. 5

$$\tan\gamma_1=\frac{\Delta v}{\Delta x} \quad \text{and} \quad \tan\gamma_2=\frac{\Delta u}{\Delta y}, \tag{14}$$

where Δu and Δv are, respectively, the relative displacements in the x and y-directions between points P_1 and P_2, which are given by expressions (12) and $\Delta v=\Delta v_0-z\frac{\partial(\Delta w)}{\partial y}$.

For small Δu and Δv relative displacements, $\tan\gamma_1\approx\gamma_1$ and $\tan\gamma_2\approx\gamma_2$. For $\Delta x,\Delta y\rightarrow 0$ the γ_1 and γ_2 distortions in the point P_1 are, respectively,

$$\gamma_1=\lim_{\Delta x,\ \Delta y\rightarrow 0}\frac{\Delta v}{\Delta x}=\frac{\partial v(x,y,z)}{\partial x} \quad \text{and} \quad \gamma_2=\lim_{\Delta x,\ \Delta y\rightarrow 0}\frac{\Delta u}{\Delta y}=\frac{\partial u(x,y,z)}{\partial y}\ . \tag{15}$$

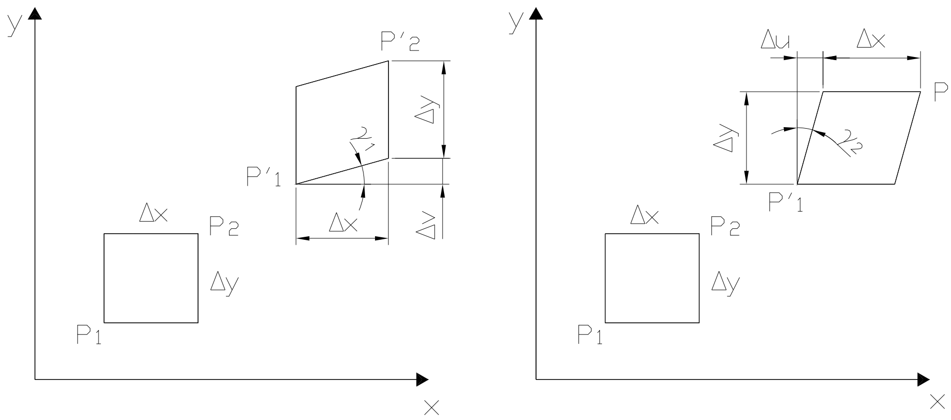
From the previous expressions and equations (6) and (13), the total distortion $\bar{\gamma}_{xy}$ is

$$\bar{\gamma}_{xy}(x,y,z)=\frac{\partial v_0(x,y)}{\partial x}+\frac{\partial u_0(x,y)}{\partial y}-2z\frac{\partial^2 w(x,y)}{\partial x\partial y}, \tag{16}$$

where

$$\bar{\gamma}_{xy}^{m}(x,y)=\frac{\partial v_0(x,y)}{\partial x}+\frac{\partial u_0(x,y)}{\partial y} \quad \text{and} \quad \bar{\gamma}_{xy}^{b}(x,y,z)=-2z\frac{\partial^2 w(x,y)}{\partial x\partial y}, \tag{17}$$

are the membrane and bending components of the total distortion $\bar{\gamma}_{xy}$.



(a) Distortion γ_1.

(b) Distortion γ_2.

Figure 5: Distortion in the plane of the reference surface.

2.4 Rigid Body Displacements

For rigid bodies, the relative displacement between any two points is constant. Hence, all points have the same displacement components and consequently the strain components are zero.

Therefore, the rigid body displacement components are obtained making zero all strain components. For the Kirchhoff plate model, it follows that

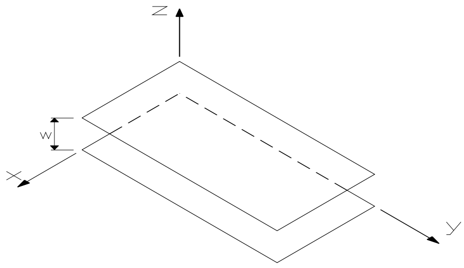
$$
\begin{array}{ll}
\text{Bending effect} & \text{Membrane effect} \\
\varepsilon_{xx}^{f}(x,y,z) = -z\dfrac{\partial^2 w(x,y)}{\partial x^2} = 0, & \varepsilon_{xx}^{m}(x,y) = \dfrac{\partial u_0(x,y)}{\partial x} = 0, \\
\varepsilon_{yy}^{f}(x,y,z) = -z\dfrac{\partial^2 w(x,y)}{\partial y^2} = 0, & \varepsilon_{yy}^{m}(x,y) = \dfrac{\partial v_0(x,y)}{\partial y} = 0, \\
\bar{\gamma}_{xy}^{f}(x,y,z) = -2z\dfrac{\partial^2 w(x,y)}{\partial x \partial y} = 0, & \gamma_{xy}^{m}(x,y) = \dfrac{\partial v_0(x,y)}{\partial x} + \dfrac{\partial u_0(x,y)}{\partial y} = 0.
\end{array}
\tag{18}
$$

The bending strain components are zero when $w(x,y) = w = cte$ or $\frac{\partial w(x,y)}{\partial x} = \frac{\partial w}{\partial x} = cte$ and $\frac{\partial w(x,y)}{\partial y} = \frac{\partial w}{\partial y} = cte$, which represent, respectively, a rigid translation of the plate in the direction z and rigid rotations about x and y-axes as illustrated in Fig. 6(a) to 6(c).

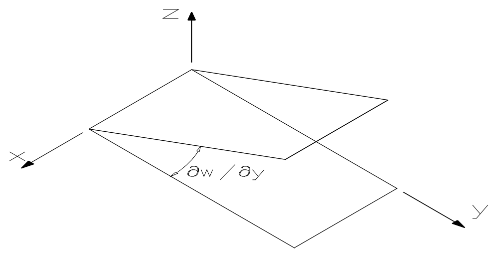
The membrane strain components are zero for $u_0 = cte$, $v_0 = cte$, $\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} = cte$, which represent, respectively, rigid translations in the x and y directions and rotation about the z axis as illustrated in Fig. 6(d) to 6(f).

2.5 Determination of Internal Loads

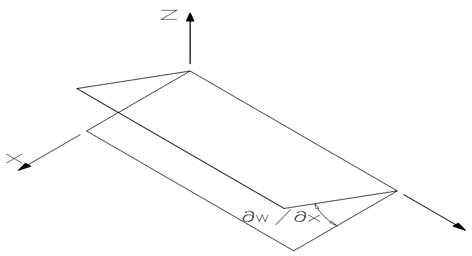
The internal loads compatible with the kinematics of the Kirchhoff plate are obtained from the internal work concept [16], which associates each strain component to its respective stress component. Based on that, the normal strain components $\varepsilon_{xx}(x,y,z)$ and $\varepsilon_{yy}(x,y,z)$ are associated to the normal stress components $\sigma_{xx}(x,y,z)$and $\sigma_{yy}(x,y,z)$, respectively. Analogously, the shear strain component $\bar{\gamma}_{xy}(x,y,z)$ is related to the shear stress component $\tau_{xy}(x,y,z)$.



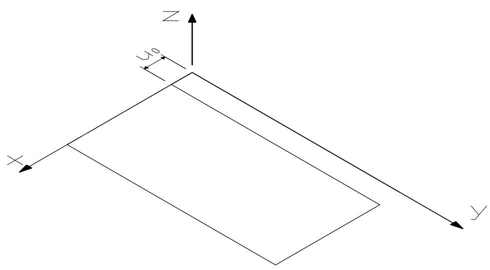
(a) Rigid translation along the z-axis.



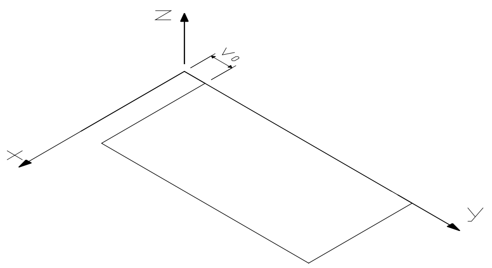
(b) Rigid rotation about the x-axis.



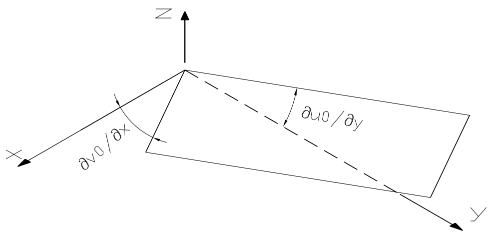
(c) Rigid rotation about the y-axis.



(d) Rigid translation along the x-axis.



(e) Rigid translation along the y-axis.



(f) Rigid rotation about the z-axis.

Figure 6: Rigid displacements for the Kirchhoff plate model.

The internal work density t_i for the Kirchhoff plate is given by [16]

$$\begin{aligned} t_i(x,y,z) &= \sigma_{xx}(x,y,z)\,\varepsilon_{xx}(x,y,z) + \sigma_{yy}(x,y,z)\,\varepsilon_{yy}(x,y,z) \\ &\quad + \tau_{xy}(x,y,z)\,\bar{\gamma}_{xy}(x,y,z). \end{aligned}$$

The SI unit for t_i and the strain components are, respectively, Nm/m^3 and m/m. Therefore, the stress components are given in N/m^2. In terms of the variational formulation, the strain and stress components represent the primal and dual variables related by t_i, which is a linear functional.

The total internal work is the sum of the internal work density for all points of the plate. As the plate is a tridimensional continuum medium, an integration over its volume is required. Hence,

$$\begin{aligned} T_i &= \int_V t_i dV = \int_V [\sigma_{xx}(x,y,z)\,\varepsilon_{xx}(x,y,z) + \sigma_{yy}(x,y,z)\,\varepsilon_{yy}(x,y,z) \\ &\quad + \tau_{xy}(x,y,z)\,\bar{\gamma}_{xy}(x,y,z)]\,dV. \end{aligned} \tag{19}$$

Substituting equations (10), (12) and (16) in the previous expression and remembering that u_0, v_0 and w depend only on the coordinates x and y, the volume integral in (19) can be decomposed in two integrals: one along the constant thickness t of the plate and the other over the area of the reference surface in the plane xy. From that decomposition, the following loads are obtained for the bending and membrane effects

$$\begin{array}{ll} \text{Bending effect} & \text{Membrane effect} \\ M_{xx}(x,y) = \int_{-t/2}^{t/2} z\,\sigma_{xx}(x,y,z)\,dz, & N_{xx}(x,y) = \int_{-t/2}^{t/2} \sigma_{xx}(x,y,z)\,dz, \\ M_{yy}(x,y) = \int_{-t/2}^{t/2} z\,\sigma_{yy}(x,y,z)\,dz, & N_{yy}(x,y) = \int_{-t/2}^{t/2} \sigma_{yy}(x,y,z)\,dz, \\ M_{xy}(x,y) = \int_{-t/2}^{t/2} z\,\tau_{xy}(x,y,z)\,dz, & N_{xy}(x,y) = \int_{-t/2}^{t/2} \tau_{xy}(x,y,z)\,dz. \end{array} \tag{20}$$

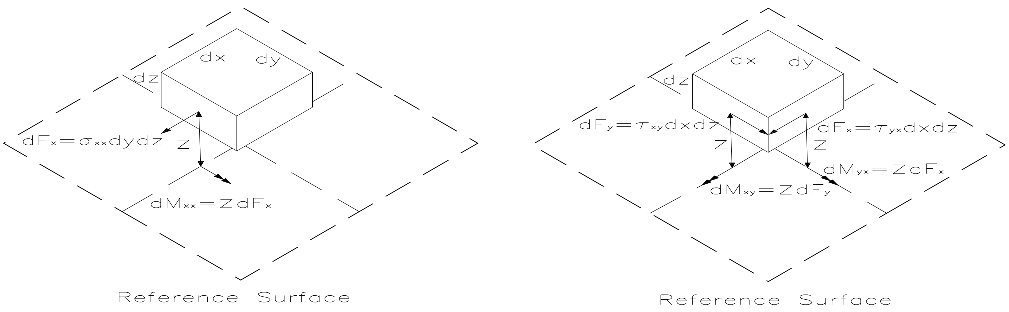
Based on the previous load definitions, the internal work may be rewritten as

$$\begin{aligned} T_i &= \int_A M_{xx}(x,y)\frac{\partial^2 w(x,y)}{\partial x^2}dA + \int_A M_{yy}(x,y)\frac{\partial^2 w(x,y)}{\partial y^2}dA \\ &\quad +2\int_A M_{xy}(x,y)\,\frac{\partial^2 w(x,y)}{\partial x \partial y}dA - \int_A N_{xx}(x,y)\,\frac{\partial u_0(x,y)}{\partial x}dA \\ &\quad -\int_A N_{yy}(x,y,z)\,\frac{\partial u_0(x,y)}{\partial y}dA \\ &\quad -\int_A N_{xy}(x,y,z)\left(\frac{\partial v_0(x,y)}{\partial x} + \frac{\partial u_0(x,y)}{\partial y}\right)dA. \end{aligned} \tag{21}$$

Fig. 7 shows an infinitesimal volume element $dxdydz$ which is located z units from the reference surface. The bending moment dM_{xx} and the twisting moments $dM_{xy} = dM_{yx}$ acting on the infinitesimal element are also illustrated in Fig. 7.

From Fig.7(a), it may be seen that $\sigma_{xx}dydz$ results in the force dF_x in the x-direction. Therefore, $z\,\sigma_{xx}dydz$ represents the bending moment dM_{xx} in the y-direction. The term $\sigma_{xx}dz$ represents the force per unit of length in the y-direction. When it is multiplied by z results in the distributed bending moment $dM_{xx} = z\,\sigma_{xx}dz$ in the x-direction. Integrating dM_{xx} along the thickness, the distributed bending moment $M_{xx}(x,y) = \int_{-t/2}^{t/2} z\,\sigma_{xx}(x,y,z)\,dz$ is obtained.

Fig. 7(b) shows the forces $dF_x = \tau_{yx}dxdz$ and $dF_y = \tau_{xy}dydz$ that represent, respectively, shear forces in the directions x and y. Multiplying these forces by z and integrating along the thickness, the twisting moments M_{xy} and M_{yx} are determined. The same interpretation may be applied to the other bending and membrane loads [15]. In this case, M_{yy} is a distribuited bending moment in the y-direction, N_{xx}



(a) Bending moment M_{xx}.

(b) Twisting moments M_{xy} and M_{yx}.

Figure 7: Interpretation of integrals along of thickness of the plate.

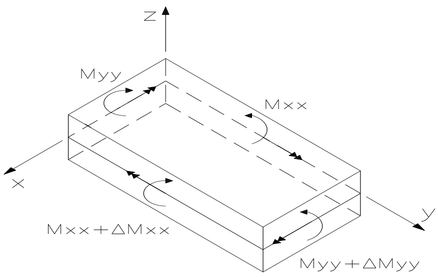
and N_{yy} are longitudinal normal forces in the x and y-directions and N_{xy} is the tangencial force in the xy-plane of the reference surface.

The following expression for the internal work results from two integrations by parts of equation (21) [15],

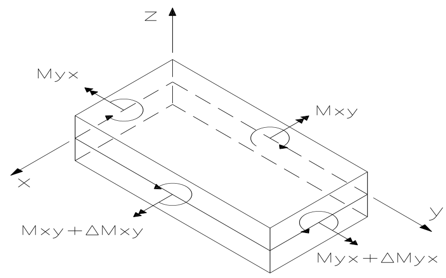
$$
\begin{aligned}
T_i \;=\; & \int_A \left(\frac{\partial^2 M_{xx}(x,y)}{\partial x^2} + 2\frac{\partial^2 M_{xy}(x,y)}{\partial x \partial y} + \frac{\partial^2 M_{yy}(x,y)}{\partial y^2} \right) w(x,y)\, dA \\
& - \oint_{\partial A} \left(\frac{\partial M_{xx}(x,y)}{\partial x} + \frac{\partial M_{xy}(x,y)}{\partial y} \right) n_x(x,y)\, w(x,y)\, d\partial A \\
& - \oint_{\partial A} \left(\frac{\partial M_{yy}(x,y)}{\partial y} + \frac{\partial M_{xy}(x,y)}{\partial x} \right) n_y(x,y)\, w(x,y)\, d\partial A \\
& + \oint_{\partial A} \left(M_{xx}(x,y)\, n_x(x,y) + M_{xy}(x,y)\, n_y(x,y) \right) \frac{\partial w(x,y)}{\partial x} d\partial A \\
& + \oint_{\partial A} \left(M_{xy}(x,y)\, n_x(x,y) + M_{yy}(x,y)\, n_y(x,y) \right) \frac{\partial w(x,y)}{\partial y} d\partial A \\
& + \int_A \left(\frac{\partial N_{xx}(x,y)}{\partial x} + \frac{\partial N_{xy}(x,y)}{\partial y} \right) u_0(x,y)\, dA \\
& + \int_A \left(\frac{\partial N_{yy}(x,y)}{\partial y} + \frac{\partial N_{xy}(x,y)}{\partial x} \right) v_0(x,y)\, dA \\
& - \oint_{\partial A} \left(N_{xx}(x,y)\, n_x(x,y) + N_{xy}(x,y)\, n_y(x,y) \right) u_0(x,y)\, d\partial A \\
& - \oint_{\partial A} \left(N_{xy}(x,y)\, n_x(x,y) + N_{yy}(x,y)\, n_y(x,y) \right) v_0(x,y)\, d\partial A,
\end{aligned}
\tag{22}
$$

where n_x and n_y are the components of the normal vector at each point of the boundary ∂A of the reference surface. The previous expression may be written as

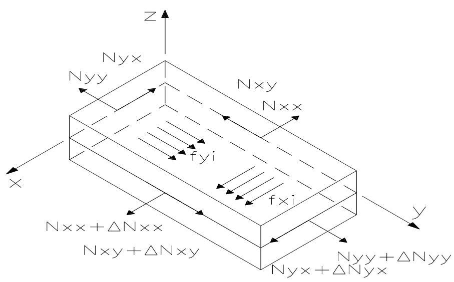
$$
\begin{aligned}
T_i \;=\; & \int_A q_i(x,y) w\,(x,y)\, dA \\
& - \oint_{\partial A} \left(Q_{xz}(x,y) n_x(x,y) + Q_{yz}(x,y) n_y(x,y)\right) w\,(x,y)\, d\partial A \\
& + \oint_{\partial A} \left(M_{xx}\,(x,y)\, n_x\,(x,y) + M_{xy}\,(x,y)\, n_y\,(x,y)\right) \frac{\partial w\,(x,y)}{\partial x} d\partial A \\
& + \oint_{\partial A} \left(M_{xy}(x,y) n_x\,(x,y) + M_{yy}\,(x,y)\, n_y\,(x,y)\right) \frac{\partial w\,(x,y)}{\partial y} d\partial A \\
& + \int_A f_{xi}\,(x,y)\, u_0\,(x,y)\, dA + \int_A f_{yi}\,(x,y)\, v_0\,(x,y)\, dA \\
& - \oint_{\partial A} \left(N_{xx}\,(x,y,z)\, n_x + N_{xy}\,(x,y)\right) n_y\,(x,y)\, u_0\,(x,y)\, d\partial A \\
& - \oint_{\partial A} \left(N_{xy}\,(x,y)\, n_x + N_{yy}\,(x,y)\right) n_y\,(x,y)\, v_0\,(x,y)\, d\partial A.
\end{aligned}
\tag{23}
$$



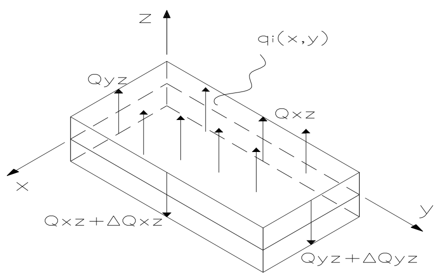
(a) Bending moments M_{xx} and M_{yy}.



(b) Twisting moments M_{xy} and M_{yx}.



(c) Normal forces N_{xx}, N_{yy} and N_{yx}.



(d) Shear forces Q_x and Q_y and distributed load q_i.

Figure 8: Internal loads for the Kirchhoff plate [18].

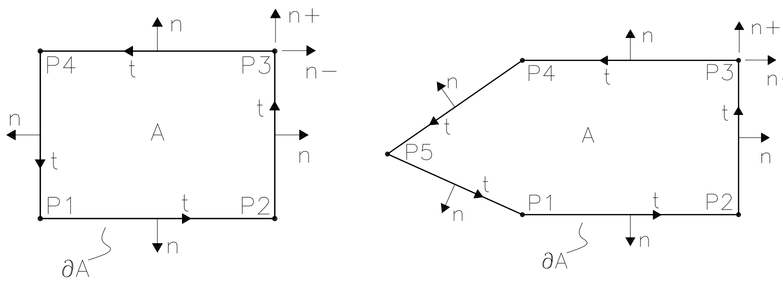
The internal loads indicated are illustrated in Figs. 8(a) to 8(d) and have the following interpretation:

- q_i represents the transversal distributed load and is normal to the reference surface;
- Q_{xz} and Q_{yz} are the shear forces in the z direction along the boundary ∂A of the reference surface;
- M_{xx} and M_{yy} are the distribuited bending moments acting on the points of the boundary of the reference surface;

- N_{xx} and N_{yy} are the normal forces in the points of the boundary ∂A;
- N_{xy} is the tangencial force acting on the points of the boundary ∂A;
- f_{xi} and f_{yi} are distributed forces tangent to the reference surface.

For non-rectangular plates (see Fig. 9(b)), it is important to represent the boundary integrands in (23) using the normal (n) and tangential (t) directions at each boundary point. Based on the manipulations indicated in [15], the final expression for the internal work valid for rectangular and non-rectangular plates is

$$\begin{aligned} T_i \;=\; & \int_A q_i(x,y) w\left(x,y\right) dA + \int_A f_{xi}\left(x,y\right) u_0\left(x,y\right) dA + \int_A f_{yi}\left(x,y\right) v_0\left(x,y\right) dA \\ & - \oint_{\partial A} V_n(x,y) w\left(x,y\right) d\partial A + \oint_{\partial A} M_{nn}(x,y) \frac{\partial w\left(x,y\right)}{\partial n} d\partial A \\ & - \oint_{\partial A} \left(N_{nn}\left(x,y\right) u_{0n}\left(x,y\right) + N_{nt}\left(x,y\right) u_{0t}\left(x,y\right)\right) d\partial A \\ & + \sum_{i=1}^{N} \left[\left(M_{nt}^{+} - M_{nt}^{-}\right) w(x,y)\right]_{P_i} . \end{aligned} \tag{24}$$

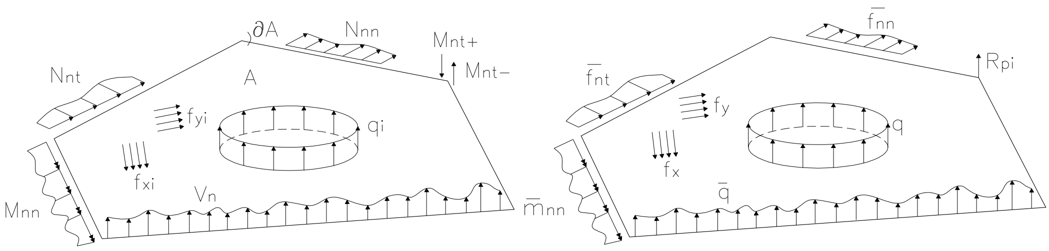


(a) Rectangular plate. (b) Polygonal plate.

Figure 9: Points of geometric discontinuities of the reference surface of the plate [18].

From the previous expression, the internal loads compatible to the kinematics of the Kirchhoff plate model are (see Fig. 10(b))

- q_i : distributed transversal force to the reference surface;
- V_n : shear force per unit of length in the z-direction of the reference surface boundary;
- M_{nn} : bending moment per unit of length in the direction t acting on the points of the reference surface boundary;
- M_{nt}^{+} and M_{nt}^{-} : transversal concentrated forces acting on the right and left sides of the discontinuous points P_i of the reference surface boundary (see Fig. 9);
- f_{xi} and f_{yi}: distributed tangential forces to the reference surface.
- N_{nn} : normal force per unit of length acting on the points of the reference surface boundary;
- N_{nt} : tangential force per unit of length acting on the points of the reference surface boundary.



(a) Internal loads. (b) External loads.

Figure 10: Internal and external loads due to the bending and membrane effects.

2.6 Determination of External Loads and Equilibrium

The external loads of the Kirchhoff plate model are those ones that can be equilibrated by the respective internal loads defined in last section. They are illustrated in Fig. 10(a) and denoted as:

- q: distributed transversal force to the reference surface;
- $\bar{q}$: distributed transversal force to the reference surface boundary;
- $\bar{m}_{nn}$: distributed moment in the t-direction of the reference surface boundary;
- R_{P_i}: concentrated transversal forces to the reference surface acting on the points of the geometric discontinuities P_i of the reference surface boundary;
- f_x: distributed tangential force in the x-direction on the reference surface;
- f_y: distributed tangential force in the y-direction on the reference surface;
- $\bar{f}_{nn}$: normal force on the reference surface boundary;
- $\bar{f}_{nt}$: tangential force on reference surface boundary.

Based on the previous load definitions, the expression of the external work may be written as

$$
\begin{aligned}
T_e \;=\; & \int_A q(x,y)w\,(x,y)\,dA + \int_A f_x(x,y)u_0\,(x,y)\,dA \\
& + \int_A f_y(x,y)v_0\,(x,y)\,dA + \int_{\partial A} \bar{q}(x,y)w\,(x,y)\,d\partial A \\
& + \oint_{\partial A} \bar{m}_{nn}\frac{\partial w\,(x,y)}{\partial n} + \oint_{\partial A} \bar{f}_{nn}(x,y)u_n\,(x,y)\,d\partial A \\
& + \oint_{\partial A} \bar{f}_{nt}(x,y)u_t\,(x,y)\,d\partial A + \sum_{i=1}^{N}\left[R_{P_i}w(x,y)\right]_{P_i}.
\end{aligned} \tag{25}
$$

The Virtual Work Principle (PVW) establishes that for a body in equilibrium the sum of the external (T_e) and internal (T_i) works must be zero to any virtual displacement $\hat{\mathbf{u}}$ compatible with the kinematics of the Kirchhoff plate model. Therefore

$$
T_e = T_i \qquad \text{for all virtual displacement } \hat{\mathbf{u}} = (u_0, v_0, w)\ . \tag{26}
$$

Substituting (24) and (25) in (26), the equilibrium Boundary Value Problem (BVP) is given by

$$\begin{cases} \dfrac{\partial^2 M_{xx}(x,y)}{\partial x^2} + \dfrac{\partial^2 M_{yy}(x,y)}{\partial y^2} + 2\dfrac{\partial^2 M_{xy}(x,y)}{\partial x \partial y} + q(x,y) = 0 & x,y \in A, \\ V_n - \bar{q} = 0 & x,y \in \partial A, \\ M_{nn}(x,y) + \bar{m}_{nn}(x,y) = 0 & x,y \in \partial A, \\ \left(M_{nt}^{+} - M_{nt}^{-}\right) - R_{P_i} = 0 & i = 1,2,...,N, \\ \dfrac{\partial N_{xx}(x,y)}{\partial x} + \dfrac{\partial N_{xy}(x,y)}{\partial y} + f_x(x,y) = 0 & x,y \in A, \\ \dfrac{\partial N_{yy}(x,y)}{\partial y} + \dfrac{\partial N_{xy}(x,y)}{\partial x} + f_y(x,y) = 0 & x,y \in A, \\ f_{nn}(x,y) - N_{nn}(x,y) = 0 & x,y \in \partial A, \\ f_{nt}(x,y) - N_{nt}(x,y) = 0 & x,y \in \partial A. \end{cases} \tag{27}$$

The boundary loads are illustrated in Fig. 11.

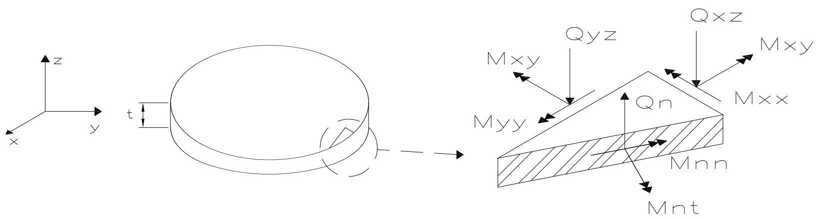


Figure 11: Boundary loads of the Kirchhoff plate model [14].

2.7 Application of the Constitutive Equation

For an elastic, linear and isotropic material, the stress and strain components are related by the Hooke's law. In this paper due to the small displacement assumption, the behaviour of the plate kinematics is also linear. Therefore, it is possible to write the stress in any point of the plate as the sum of the bending and membrane stresses, i.e., $\sigma = \sigma^m + \sigma^b$.

The bending stress and strain components are related by the Hooke's law as [14]

$$\begin{aligned} \sigma_{xx}^b(x,y,z) &= \frac{E}{1-\nu^2}\Big(\varepsilon_{xx}^b(x,y,z) + \nu\,\varepsilon_{yy}^b(x,y,z)\Big), \\ \sigma_{xx}^b(x,y,z) &= \frac{E}{1-\nu^2}\Big(\varepsilon_{xx}^b(x,y,z) + \nu\,\varepsilon_{yy}^b(x,y,z)\Big), \\ \tau_{xy}^b(x,y,z) &= \frac{E(1-\nu)}{2(1-\nu^2)}\bar{\gamma}_{xy}^b(x,y,z), \end{aligned} \tag{28}$$

where E and ν are the Young's modulus and Poisson's coefficient of the material. The Hooke's law for the membrane effect is similar to (28) just changing the superscript b to m.

Substituting the strain components given in (10), (12) and (17) in the Hooke's law, the following expressions for the bending and membrane stresses are obtained

$$
\begin{aligned}
\sigma_{xx}^{b}(x,y) &= -z\frac{E}{1-\nu^2}\left(\frac{\partial^2 w(x,y)}{\partial x^2}+\nu\frac{\partial^2 w(x,y)}{\partial y^2}\right),\\
\sigma_{yy}^{b}(x,y) &= -z\frac{E}{1-\nu^2}\left(\frac{\partial^2 w(x,y)}{\partial y^2}+\nu\frac{\partial^2 w(x,y)}{\partial x^2}\right),\\
\tau_{xy}^{b}(x,y) &= -z\frac{E(1-\nu)}{1-\nu^2}\frac{\partial^2 w(x,y)}{\partial x\partial y},\\
\tau_{xy}^{m}(x,y) &= \frac{E(1-\nu)}{2(1-\nu^2)}\left(\frac{\partial v_0(x,y)}{\partial x}+\frac{\partial u_0(x,y)}{\partial y}\right)\\
\sigma_{xx}^{m}(x,y) &= \frac{E}{1-\nu^2}\left(\frac{\partial u_0(x,y)}{\partial x}+\nu\frac{\partial v_0(x,y)}{\partial y}\right),\\
\sigma_{yy}^{m}(x,y) &= \frac{E}{1-\nu^2}\left(\frac{\partial v_0(x,y)}{\partial y}+\nu\frac{\partial u_0(x,y)}{\partial x}\right).
\end{aligned}
\tag{29}
$$

Substituting now the previous expressions in definition of the internal loads, the bending and twisting moments and the shear and normal forces are given in terms of the displacement component as

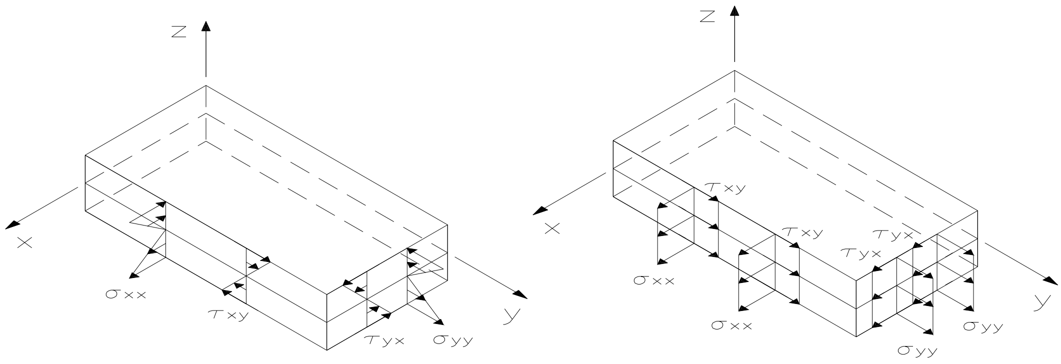
$$
\begin{aligned}
M_{xx}(x,y) &= -D\left(\frac{\partial^2 w(x,y)}{\partial x^2}+\nu\frac{\partial^2 w(x,y)}{\partial y^2}\right),\\
M_{yy}(x,y) &= -D\left(\frac{\partial^2 w(x,y)}{\partial y^2}+\nu\frac{\partial^2 w(x,y)}{\partial x^2}\right),\\
M_{xy}(x,y) &= -D(1-\nu)\frac{\partial^2 w(x,y)}{\partial x\partial y},\\
N_{xx}(x,y) &= T\left(\frac{\partial u_0(x,y)}{\partial x}+\nu\frac{\partial v_0(x,y)}{\partial y}\right),\\
N_{yy}(x,y) &= T\left(\frac{\partial v_0(x,y)}{\partial y}+\nu\frac{\partial u_0(x,y)}{\partial x}\right),\\
N_{xy}(x,y) &= T(1-\nu)\left(\frac{\partial v_0(x,y)}{\partial x}+\frac{\partial u_0(x,y)}{\partial y}\right),\\
Q_{x}(x,y) &= -D\left(\frac{\partial^3 w(x,y)}{\partial x^3}+\frac{\partial^3 w(x,y)}{\partial x\partial y^2}\right),\\
Q_{y}(x,y) &= -D\left(\frac{\partial^3 w(x,y)}{\partial y^3}+\frac{\partial^3 w(x,y)}{\partial x^2\partial y}\right),
\end{aligned}
\tag{30}
$$

where $D=\frac{Et^3}{12(1-\nu^2)}$ and $T=\frac{Et}{(1-\nu^2)}$ are the bending and membrane stiffnesses of the Kirchhoff plate.

The derivatives presented in (29) may be expressed in terms of the loads given in (30) and the stress components reduce to

$$
\begin{aligned}
\sigma_{xx}^{f}(x,y) &= \frac{M_{xx}(x,y)}{t^3/12}z, & \sigma_{xx}^{m}(x,y) &= \frac{N_{xx}(x,y)}{t},\\
\sigma_{yy}^{f}(x,y) &= \frac{M_{yy}(x,y)}{t^3/12}z, & \sigma_{yy}^{m}(x,y) &= \frac{N_{yy}(x,y)}{t},\\
\tau_{xy}^{f}(x,y) &= \frac{M_{xy}(x,y)}{t^3/12}z, & \tau_{xy}^{m}(x,y) &= \frac{N_{xy}(x,y)}{t}.
\end{aligned}
\tag{31}
$$

These stress distributions are illustrated in Fig. 12. Note that the bending stresses vary linearly while the membrane stresses are constants according to the kinematics.



(a) Bending normal and shear stress distributions.

(b) Membrane normal and shear stress distributions.

Figure 12: Bending and membrane stress distributions in the Kirchhoff model [18].

Substituting the loads expressions (30) in the differential equations of the BVP (27), the differential equations in terms of the displacements for the bending and membrane cases are, respectively,

$$D\left(\frac{\partial^4 w\left(x,y\right)}{\partial x^4}+\frac{\partial^4 w(x,y)}{\partial y^4}+2\frac{\partial^4 w\left(x,y\right)}{\partial x^2\partial y^2}\right)=q\left(x,y\right). \tag{32}$$

$$\begin{aligned} &T\left[\frac{(1-\nu)}{2}\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial x\partial y}\right]=-f_x,\\ &T\left[\frac{(1-\nu)}{2}\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial^2 u_0}{\partial x\partial y}\right]=-f_y. \end{aligned} \tag{33}$$

The essential boundary conditions for the bending problems are expressed in terms of the transversal displacement w and the rotations $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. The natural boundary conditions are given in terms of the bending and twisting moments and the shear forces. For the membrane problem, the essential and natural boundary conditions are given in terms of the displacements components (u_0, v_0) and the normal forces, respectively.

3 Finite Element Approximation

In this section the finite element approximation of the Kirchhoff plate is developed applying a procedure with the following main steps: determination of the strong form expressed in operator notation and indication of the solution domain; determination of the weak form related to the strong form using the Weighted Residual Method; approximation of the weak form by the Galerkin Method; use of interpolation functions with compact support to obtain the plate finite elements.

3.1 Strong Form

The differential equations or strong forms (32) and (33) in terms of the displacements components (w, u_0, v_0) may be expressed in an operator form as

$$A^b u^b = f^b \quad \text{and} \quad \mathbf{A}^m\mathbf{u}^m = \mathbf{f}^m,$$

where the differential operators A^b and $\mathbf{A}^m$ are given, respectively, by

$$A^b = D\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}\right), \tag{34}$$

$$\mathbf{A}^m = T\begin{bmatrix} \frac{(1-\nu)}{2}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{(1-\nu)}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{bmatrix}, \tag{35}$$

and $u^b = w(x,y)$, $f^b = q(x,y)$, $\mathbf{u}^m = \{\ u_0(x,y) \quad v_0(x,y)\ \}^T$ and $\mathbf{f}^m = \{\ -f_x(x,y) \quad -f_y(x,y)\ \}^T$.

The classical solutions of the bending and membrane differential equations belong to the domains of the operators A^b and $\mathbf{A}^m$, i.e.,

$$\begin{aligned} D_{A^b} &= \left\{w(x,y) \in C^4(x,y) \text{ and satisfies the boundary conditions}\right\}, \\ D_{A^m} &= \left\{u_0(x,y), v_0(x,y) \in C^2(x,y) \text{ and satisfy the boundary conditions}\right\}. \end{aligned}$$

3.2 Weak Form

It is possible to require less regularity of the plate BVP solution by considering the weak forms associated to the strong forms (32) and (33). For that purpose, equations (32) and (33) are multiplied by appropriate test functions that satisfy the homogeneous essential boundary conditions of the problem. After integration by parts, the weak forms of the bending and membrane effects are [3–9,14,18]

$$\begin{aligned} &D\int_A \left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \nu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\right) + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + 2(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\right] dA \\ &- \oint_{\partial A}\left(Q_n + \frac{\partial M_{nt}}{\partial t}\right) v\, d\partial A - \oint_{\partial A} M_{nn}\frac{\partial v}{\partial n} d\partial A + \sum_{i=1}^{N}\left[\left(M_{nt}^+ - M_{nt}^-\right) v\right]_{P_i} \\ &- \int_A q\, v\, dA = 0\,. \end{aligned} \tag{36}$$

$$\begin{aligned} &T\int_A \left[\frac{\partial u_0}{\partial x}\frac{\partial v}{\partial x} + \nu\frac{\partial v_0}{\partial y}\frac{\partial v}{\partial x} + \frac{(1-\nu)}{2}\left(\frac{\partial v_0}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u_0}{\partial y}\frac{\partial v}{\partial y}\right)\right] dA \\ &= \oint_{\partial A}\left[N_{xx}n_x + N_{xy}n_y\right] v\, d\partial A - \int_A f_x\, v\, dA, \end{aligned} \tag{37}$$

$$\begin{aligned} &T\int_A \left[\frac{\partial v_0}{\partial y}\frac{\partial v}{\partial y} + \nu\frac{\partial u_0}{\partial x}\frac{\partial v}{\partial y} + \frac{(1-\nu)}{2}\left(\frac{\partial v_0}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial y}\frac{\partial v}{\partial x}\right)\right] dA \\ &= \oint_{\partial A}\left[N_{xy}n_x + N_{yy}n_y\right] v\, d\partial A - \int_A f_y\, v\, dA. \end{aligned} \tag{38}$$

It may be observed that the bending and membrane weak form solutions require less regularity than the respective strong forms. For the bending problem, the solution belongs to the $C^1(x,y)$ class of functions while the solution of membrane weak form belongs to $C^0(x,y)$.

3.3 Approximation of the Weak Form

The approximated solution $w_n(x,y)$ of the bending weak form is written by the following linear combination

$$w_n(x,y) = \sum_{i=1}^{n} a_i \phi_i(x,y), \tag{39}$$

where a_i are the unknown coefficients and $\{\phi_i\}_{i=1}^n$ are C^1 are the interpolation functions. For the Kirchhoff plate, they are given by Hermite polynomials [14, 15].

Using the Galerkin method, the test function $v\,(x,y)$ is interpolated using the same set of interpolation functions and

$$v_n\,(x,y) = \sum_{j=1}^{n} b_j \phi_j\,(x,y)\,. \tag{40}$$

Substituting w_n and v_n in (36), the following system of equations is obtained

$$\left[\mathbf{K}^b\right]\{\mathbf{a}\} = \left\{\mathbf{f}^b\right\}, \tag{41}$$

where $\left[\mathbf{K}^b\right]$ and $\left[\mathbf{f}^b\right]$ are the bending stiffness matrix and load vector, respectively. Their coefficients are for $i,j = 1,\ldots,n$

$$\begin{aligned} K_{ij}^b &= D \int_A \left[\frac{\partial^2\phi_i}{\partial x^2}\frac{\partial^2\phi_j}{\partial x^2} + \nu\left(\frac{\partial^2\phi_i}{\partial x^2}\frac{\partial^2\phi_j}{\partial y^2} + \frac{\partial^2\phi_i}{\partial y^2}\frac{\partial^2\phi_j}{\partial x^2}\right)\right. \\ &\left. + \frac{\partial^2\phi_i}{\partial y^2}\frac{\partial^2\phi_j}{\partial y^2} + 2\,(1-\nu)\,\frac{\partial^2\phi_i}{\partial x\partial y}\frac{\partial^2\phi_j}{\partial x\partial y}\right] dA, \end{aligned} \tag{42}$$

$$\begin{aligned} f_j^b &= \oint_{\partial A}\left(Q_n + \frac{\partial M_{nt}}{\partial t}\right)\phi_j \, d\partial A + \int_{\partial A} M_{nn}\frac{\partial \phi_j}{\partial n} d\partial A \\ &- \sum_{k,j=1}^{N}\left[\left(M_{nt}^+ - M_{nt}^-\right)\phi_j\right]_{P_k} + \int_A q\,\phi_j\,dA. \end{aligned} \tag{43}$$

The bending stiffness matrix may be written in the standard form

$$\left[\mathbf{K}^b\right] = \frac{t^3}{12}\left[\mathbf{B}^b\right]^T[\mathbf{D}]\left[\mathbf{B}^b\right], \tag{44}$$

where

$$[\mathbf{B}_i^b] = \begin{bmatrix} -\dfrac{\partial^2\phi_1}{\partial x^2} & \ldots\ldots. & -\dfrac{\partial^2\phi_n}{\partial x^2} \\ -\dfrac{\partial^2\phi_1}{\partial y^2} & \ldots\ldots. & -\dfrac{\partial^2\phi_n}{\partial y^2} \\ -\dfrac{\partial^2\phi_1}{\partial x\partial y} & \ldots\ldots. & -\dfrac{\partial^2\phi_n}{\partial x\partial y}, \end{bmatrix}, \quad [\mathbf{D}] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}.$$

The membrane displacements are approximated as

$$u_{0n}\,(x,y) = \textstyle\sum_{i=1}^n c_i\varphi_i\,(x,y)\,, \quad v_{0n}\,(x,y) = \sum_{i=1}^n d_i\varphi_i\,(x,y)\,, \tag{45}$$

where φ_i are the C^0 interpolation functions and c_i and d_i are the unknown coefficients of the linear combinations. Using the Galerkin method, the test functions u_{0n} and v_{0n} are interpolated using the functions φ_i and the following system of equations is obtained [14, 15]

$$\begin{bmatrix} \left[K^{11}\right]_m & \left[K^{12}\right]_m \\ \left[K^{12}\right]_m^T & \left[K^{22}\right]_m \end{bmatrix}\left\{\begin{array}{c} \{c\} \\ \{d\} \end{array}\right\} = \left\{\begin{array}{c} \left\{f^1\right\}_m \\ \left\{f^2\right\}_m \end{array}\right\}, \tag{46}$$

where for $i, j = 1, \ldots, n$

$$
\begin{aligned}
K_{ij}^{11} &= T \int_A \left[\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{(1-\nu)}{2} \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} \right] dA, \\
K_{ij}^{12} &= T \int_A \left[\nu \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} + \frac{(1-\nu)}{2} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} \right] dA, \\
K_{ij}^{22} &= T \int_A \left[\frac{(1-\nu)}{2} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} \right] dA,
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
f_j^1 &= \oint_{\partial A} \left[N_{xx} n_x + N_{xy} n_y \right] \varphi_j \, d\partial A - \int_A f_x \, \varphi_j \, dA, \\
f_j^2 &= \oint_{\partial A} \left[N_{xy} n_x + N_{yy} n_y \right] \varphi_j \, d\partial A - \int_A f_y \, \varphi_j \, dA.
\end{aligned}
\tag{48}
$$

3.4 Plate Finite Elements

The FEM may be defined as the Galerkin method where the interpolation functions have compact support. This feature allows to define local approximations in terms of the finite elements which are assembled conveniently to generate the global approximation.

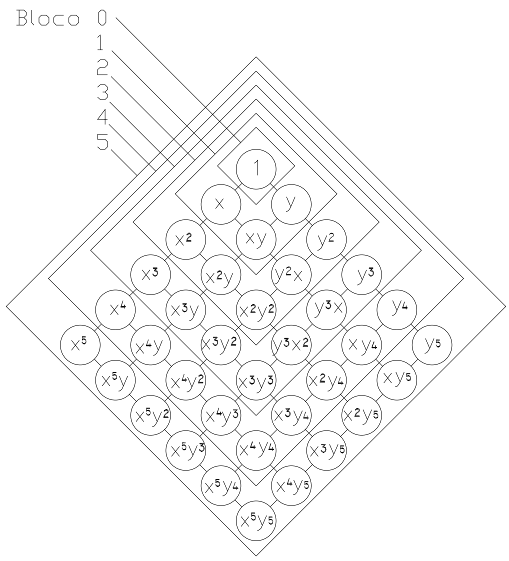


Figure 13: Pascal's triangle for quadrangular elements [16].

The minimum regularity required for the shape functions to be used in the calculation of the coefficients of the bending stiffness matrix is 1 as may be observed from equation (42). This regularity implies that the displacement w and their partial derivatives $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial^2 w}{\partial x \partial y}$ are continuous between the common edge of two finite elements. But the strain and stress components are discontinuous since their expressions involve second derivatives.

The determination of $C^1(\Omega)$ local interpolation functions for $\Omega \subset R^2$ is not simple. For quadrangular elements, the following procedure is presented in [16] to generate C^m interpolation functions. For the Pascal's triangle illustrated in Fig. 13, consider the squares $1, 3, 5, \ldots m$ with $(m+1)^2$ independent monomials which constitute the complete polynomials of odd degree. The degree of each monomial in the

m_{th} square corresponds to the derivative order to be prescribed at the nodes of the 4 node quadrangular element. The local interpolation function will contain all monomials of the square $2m+1$.

For example, the square $m=1$ contains the monomials $1, x, y, xy$. Hence, in the four nodes of the quadrangular element there will be coefficients which correspond to the values of w, w_x, w_y and w_{xy}. The local interpolation functions will contain all the monomials of the square $2m+1=3$. The shape functions obtained from the tensorial product of one-dimensional cubic Hermite polynomials have the required properties for C^1 approximation [14, 15]. The quadrangular element is illustrated in Fig. 14.

The local approximation $w_n^{(e)}$ for the transversal displacement is

$$w_n^{(e)}(\xi,\eta) = \sum_{i=1}^{16} a_i \phi_i^{(e)}(\xi,\eta),$$

where $\phi_i^{(e)}$ are the Hermite interpolation functions and a_i the unknown coefficients. The coefficients a_1, a_5, a_9 and a_{13} correspond to the values of w; a_2, a_6, a_{10} and a_{14} to w_x; a_3, a_7, a_{11} and a_{15} to w_y; and a_4, a_8, a_{12} and a_{16} to w_{xy}.

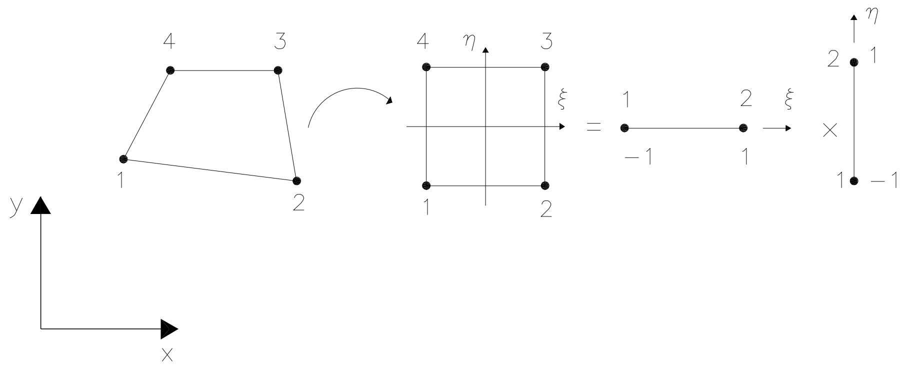


Figure 14: Quadrilateral element.

The local bending stiffness matrix is given by

$$\left[\mathbf{K}_b^{(e)}\right] = \frac{t^3}{12}\int_{A^{(e)}} \left[\mathbf{B}^b\right]^T [\mathbf{D}] \left[\mathbf{B}^b\right] \det[\mathbf{J}]\, d\xi d\eta.$$

where

$$\left[\mathbf{B}^f\right] = \left[\ [\mathbf{B}_1^b] \quad [\mathbf{B}_2^b] \quad [\mathbf{B}_3^b] \quad [\mathbf{B}_4^b]\ \right].$$

The nodal $[\mathbf{B}_i^b]$ matrices have the general expression

$$\left[\mathbf{B}_i^f\right] = \begin{bmatrix} -\dfrac{\partial^2\phi_1^{(i)}}{\partial x^2} & -\dfrac{\partial^2\phi_2^{(i)}}{\partial x^2} & -\dfrac{\partial^2\phi_3^{(i)}}{\partial x^2} & -\dfrac{\partial^2\phi_4^{(i)}}{\partial x^2} \\ -\dfrac{\partial^2\phi_1^{(i)}}{\partial y^2} & -\dfrac{\partial^2\phi_2^{(i)}}{\partial y^2} & -\dfrac{\partial^2\phi_3^{(i)}}{\partial y^2} & -\dfrac{\partial^2\phi_4^{(i)}}{\partial y^2} \\ -\dfrac{\partial^2\phi_1^{(i)}}{\partial x\partial y} & -\dfrac{\partial^2\phi_2^{(i)}}{\partial x\partial y} & -\dfrac{\partial^2\phi_3^{(i)}}{\partial x\partial y} & -\dfrac{\partial^2\phi_4^{(i)}}{\partial x\partial y} \end{bmatrix}.$$

The extension to triangular elements is presented in [14, 15]. The local shape functions for the membrane effect are the standard Serendipty functions presented in the literature for the 4 node quadrangular element [19, 20].

4 Final Comments

This paper presented procedures for the variational formulation and finite element approximation of mechanical models with application to the Kirchhoff plate. These procedures have been used to teach many mechanical models including bar, torsion, beam (Euller-Bernouilli and Timoshenko models), plane stress, elastic solid, Newtonian fluid and Reissner-Mindlin plate.

In addition to the systematic approach, the procedures make clear all the hypotheses and steps to obtain the formulation and approximation of mechanical models. These features allow to acquire a strong background and consequently the confident use of simulation software.

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